1. Which of the following sets is linearly independent? If the set is dependent, give an example of a nontrivial linear combination that equals zero. Yo must justify your answer.

(a) (3 points) $\{[1 \quad 6]^T, [-3 \quad 5]^T\}$

Suppose $a\begin{bmatrix}1\\6\end{bmatrix} + b\begin{bmatrix}-3\\5\end{bmatrix} = 0$, i.e. $\begin{matrix} a - 3b = 0 \\ 6a + 5b = 0 \end{matrix}$.

We solve this homogeneous linear system:

$$\begin{bmatrix}1 & -3\\6 & 5\end{bmatrix} \sim \begin{bmatrix}1 & -3\\0 & 23\end{bmatrix} \sim \begin{bmatrix}1 & -3\\0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Since the rank of the coefficient matrix is $2 = \#$ of variables, the homogeneous linear system is uniquely solvable, i.e. the only solution is $a = 0 = b$. Hence, the set is lin. independent.

Other solution: $A = \begin{bmatrix}1 & -3\\6 & 5\end{bmatrix}$ has determinant $5 + 18 = 23$, hence is invertible (§1.5). By the Invertible Matrix Theorem, the columns of A are lin. indep.

(b) (3 points) $\{[1 \quad -1 \quad 0]^T, [3 \quad 2 \quad -1]^T, [5 \quad 0 \quad -1]^T\}$. We proceed as in (a), and get a homogeneous linear system whose coefficient matrix is

$$\begin{bmatrix}1 & 3 & 5\\-1 & 2 & 0\\0 & -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 5\\0 & 5 & 5\\0 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 5\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}.$$

Since the rank $= 2 < 3 = \#$ of variables, the general solution depends on 1 parameter

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}-2t\\-t\\t\end{bmatrix}, \quad t \in \mathbb{R} \text{ arbitrary}.$$

Because the homogeneous lin. system has a non-trivial solution, the vectors are linearly dependent. A non-trivial linear dependence relation is obtained by any non-trivial solution, e.g. for $t = 1$:

$$-2\begin{bmatrix}1\\-1\\0\end{bmatrix} - \begin{bmatrix}3\\2\\-1\end{bmatrix} + \begin{bmatrix}5\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

2. (a) **(3 points)** Verify that the given vectors span $\mathbb{R}^3$:

$$[-5 \;\; -2 \;\; -2]^T, [7 \;\; -9 \;\; 3]^T, [4 \;\; -8 \;\; 9]^T, [8 \;\; 4 \;\; 7]^T.$$

Let $[a \; b \; c]^T$ be an arbitrary vector of $\mathbb{R}^3$. We have to show that there exist x, y, z, w such that

$$x\begin{bmatrix}-5\\-2\\-2\end{bmatrix} + y\begin{bmatrix}7\\-9\\3\end{bmatrix} + z\begin{bmatrix}4\\-8\\9\end{bmatrix} + w\begin{bmatrix}8\\4\\7\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}, \text{ i.e. } \begin{matrix}-5x+7y+4z+8w=a\\-2x-9y-8z+4w=b\\-2x+3y+9z+7w=c\end{matrix}$$

augmented matrix

$$\begin{bmatrix}-5 & 7 & 4 & 8 & a\\-2 & -9 & -8 & 4 & b\\-2 & 3 & 9 & 7 & c\end{bmatrix} \underset{R_1 \to R_1-2R_3}{\sim} \begin{bmatrix}-1 & 1 & -14 & -6 & a-2c\\2 & 9 & 8 & -4 & -b\\2 & -3 & -9 & -7 & -c\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & -1 & 14 & 6 & -a+2c\\0 & 11 & -20 & -16 & 2a-b-4c\\0 & -1 & -37 & -19 & 2a-5c\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 14 & 6 & -a+2c\\0 & 1 & 37 & 19 & 5c-2a\\0 & 11 & -20 & -16 & 2a-b-4c\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & -1 & 14 & 6 & -a+2c\\0 & 1 & 37 & 19 & 5c-2a\\0 & 0 & -427 & -225 & *\end{bmatrix}$$

It follows that this matrix has rank 3; hence the linear system is solvable, i.e. the 4 vectors span $\mathbb{R}^3$.

Note: It is enough to show that the first 3 vectors span $\mathbb{R}^3$. This can be done by proving that the 3×3 whose columns are the first 3 vectors is invertible.

(b) **(1 points)** Do the vectors form a basis of $\mathbb{R}^3$?

No, since a basis of $\mathbb{R}^3$ has exactly 3 vectors.

(Note: But one can delete one of the vectors, e.g. the last one, and obtain a basis of $\mathbb{R}^3$)

3. Show that $U = \{[\,3a+4b \quad 7a-b \quad a+b \quad -3b\,]^T \in \mathbb{R}^4 : a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$. Find a basis and $\dim U$.

Every vector in U can be writen as

$$\begin{bmatrix} 3a+4b \\ 7a-b \\ a+b \\ -3b \end{bmatrix} = a\begin{bmatrix} 3 \\ 7 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 4 \\ -1 \\ 1 \\ -3 \end{bmatrix}$$

Hence U is the set of linear combinations of $\begin{bmatrix} 3 \\ 7 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -1 \\ 1 \\ -3 \end{bmatrix}$. Thus,

$U = \operatorname{span}\{[3,7,1,0]^T, [4,-1,1,-3]^T\}$ is a subspace of $\mathbb{R}^4$

(§4.1, Th. 1). We already have a spanning set of U, namely

$$B = \left\{ \begin{bmatrix} 3 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \\ -3 \end{bmatrix} \right\}.$$

We check, if B is linearly independent: Suppose

$$a\begin{bmatrix} 3 \\ 7 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 4 \\ -1 \\ 1 \\ -3 \end{bmatrix} = 0 \quad , \text{ i.e. } \begin{bmatrix} 3a+4b \\ 7a-b \\ a+b \\ -3b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From the last component we get $b = 0$, and then $a = 0$ follows.

Hence B is lin. independent, and therefore a basis of U.

We have $\dim U = 2$ since the basis B of U has 2 elements

(Marking: subspace + spanning 3, lin indep 2, dim 1)

4. (6 points) Find a basis and the dimension of ~~A~~ null A where

$$A = \begin{bmatrix} 2 & 3 & -3 & 7 & -1 \\ 2 & 0 & 2 & 4 & -5 \end{bmatrix}.$$

The null space null A is the set of solutions of the homogeneous linear system whose coefficient matrix is A.

$$A = \begin{bmatrix} 2 & 3 & -3 & 7 & -1 \\ 2 & 0 & 2 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -3 & 7 & -1 \\ 0 & -3 & 5 & -3 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 2 & 4 & -5 \\ 0 & 3 & -5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 & -5/2 \\ 0 & 1 & -5/3 & 1 & 4/3 \end{bmatrix}$$

equivalent new system

$$x_1 + x_3 + 2x_4 - 5/2\, x_5 = 0$$
$$x_2 - 5/3\, x_3 + x_4 + 4/3\, x_5 = 0$$

general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s - 2t - 5/2\, u \\ 5/3\, s - t - 4/3\, u \\ s \\ t \\ u \end{bmatrix} = s \begin{bmatrix} -1 \\ 5/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -5/2 \\ -4/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus null(A) = span(B), for

$$B = \left\{ \begin{bmatrix} 1 \\ -5/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ -4/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To check that B is linearly independent: Suppose

$$s \begin{bmatrix} 1 \\ -5/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -5/2 \\ -4/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We get $s = 0 = t = u$ by considering the last 3 components.

Hence B is lin. indep, and therefore a basis of null(A).

We have dim null(A) = 3.

5. Let U and V be any subspaces of $\mathbb{R}^n$.

(a) **(3 points)** Show that $U \cap V = \{x \in \mathbb{R}^n : x \in U \text{ and } x \in V\}$ is a subspace of $\mathbb{R}^n$.

We check the 3 conditions defining a subspace (see the definition on p.182)

(S1): The zero vector lies in U and in V since both are subspaces. Therefore the zero vector lies in $U \cap V$.

(S2) Suppose $X \in U \cap V$ and $Y \in U \cap V$. By definition of $U \cap V$, this means $X \in U$ and $X \in V$, and also $Y \in U$ and $Y \in V$. The axiom (S2) applied for U and for V then shows $X+Y \in U$ and $X+Y \in V$. Hence $X+Y \in U \cap V$

(S3) Suppose $X \in U \cap V$ and $s \in \mathbb{R}$. Thus $X \in U$ and $X \in V$. By (S3) for U and for V, we get $sX \in U$ and $sX \in V$. Hence $sX \in U \cap V$.

(b) **(3 extra points)** Show that $U + V = \{X \in \mathbb{R}^n : X = X_U + X_V \text{ for some } X_U \in U \text{ and some } X_V \in V\}$ is also a subspace of $\mathbb{R}^n$.

We check the 3 conditions defining a subspace

(S1) The zero vector 0 lies in U and in V, hence $0 = 0 + 0$ can be written as a sum of some vector in U and some vector in V. Thus $0 \in U+V$

(S2) Let $X \in U+V$ and $Y \in U+V$. Thus, $X = X_U + X_V$ and $Y = Y_U + Y_V$ for vectors $X_U, Y_U \in U$ and $X_V, Y_V \in V$. By (S2) for U and V we get $X_U + Y_U \in U$ and $X_V + Y_V \in V$. Hence

$$X + Y = (X_U + Y_U) + (X_V + Y_V)$$

is a sum of a vector in U and a vector in V. Thus (S2) holds for $U+V$.

(S3) Let $X \in U+V$ and let $r \in \mathbb{R}$. Thus, $X = X_U + X_V$ for $X_U \in U$ and $X_V \in V$. Then $rX_U \in U$ and $rX_V \in V$ by (S3) for U and V. Hence $rX = (rX_U) + (rX_V)$ is a sum of a vector in U and a vector in V and therefore $rX \in U+V$.