1. (3 points) Find A if $A^T - 3\begin{bmatrix} i & -1 & 4 \\ 7 & 3 & \alpha \end{bmatrix} = (2+i)\begin{bmatrix} 4-i & 7 & 2i \\ 3 & 1+i & 9 \end{bmatrix}$, where α is the last digit of your student number.

$$A^T = 3\begin{bmatrix} i & -1 & 4 \\ 7 & 3 & \alpha \end{bmatrix} + \begin{bmatrix} (2+i)(4-i) & (2+i)7 & (2+i)(2i) \\ (2+i)3 & (2+i)(1+i) & (2+i)9 \end{bmatrix} =$$

$$= \begin{bmatrix} 3i & -3 & 12 \\ 21 & 9 & 3\alpha \end{bmatrix} + \begin{bmatrix} 9+2i & 14+7i & -2+4i \\ 6+3i & 1+3i & 18+9i \end{bmatrix} =$$

$$= \begin{bmatrix} 9+5i & 11+7i & 10+4i \\ 27+3i & 10+3i & (3\alpha+18)+9i \end{bmatrix}$$

$$\text{So } A = \begin{bmatrix} 9+5i & 27+3i \\ 11+7i & 10+3i \\ 10+4i & (3\alpha+18)+9i \end{bmatrix}$$

My answer: $\begin{bmatrix} 9+5i & 27+3i \\ 11+7i & 10+3i \\ 10+4i & (3\alpha+18)+9i \end{bmatrix}$

2. In each case either show that the statement is true or give an example with concrete numbers showing that it is false. Assume that a linear system is given with augmented matrix A and coefficient matrix C. Let R be the reduced row-echelon form of A.

(a) **(1 point)** A and R have the same size.

True, since row operations do not change the size of the matrix

(b) **(2 points)** If there is more than one solution, R must have a row of zeros.

False. For example, the linear system $x+y=0$ has $A=[1\ 1\ 0]=R$, has infinitely many solutions $x=-t$, $y=t$, t a free parameter, but R does not have a row of zeros

(c) **(2 points)** If there are more variables than equations, there are infinitely many solutions.

False. The linear system $\begin{pmatrix} x+y+z=1 \\ x+y+z=-1 \end{pmatrix}$ has 2 equations, 3 variables, but is not solvable.

(d) **(1 point)** If the system has a solution, rank $(A) =$ rank (C).

True (see my lecture of May 14)

(e) **(1 point)** rank $(A) \leq 1 +$ rank (C).

True, see my lecture of May 14.

3. (5 points) Exercise 13 of section 1.1.

(a) Let x_i, $i=1,2,3$, be the number of pills of brand i. Then

Vitamin A : $x_1 + x_2 = 5$

Vitamin B : $2x_1 + x_2 + x_3 = 13$

Vitamin C : $4x_1 + 3x_2 + x_3 = 23$

The corresponding linear system has augmented matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 5 \\ 2 & 1 & 1 & 13 \\ 4 & 3 & 1 & 23 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8-t \\ t-3 \\ t \end{bmatrix}$, $t \in \mathbb{R}$ arbitrary.

Since every $x_i \in \mathbb{N} = \{0,1,2,\dots\}$ it follows that $t \leq 8$ (because of $x_1 \in \mathbb{N}$), $t \geq 3$ (because of $x_2 \in \mathbb{N}$) and $t \in \mathbb{N}$ (because of $x_3 \in \mathbb{N}$). Thus, the possible combinations are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8-t \\ t-3 \\ t \end{bmatrix} \text{ for } t = 3,4,5,6,7,8.$$

(b) The cost of a combination in (a) is

$$0.9x_1 + 0.6x_2 + 1.5x_3 = 0.9(8-t) + 0.6(t-3) + 1.5t$$
$$= 7.2 - 1.8 + t(1.5 + 0.6 - 0.9) = 5.4 + 1.2t$$

This an increasing function with slope 1.2. Its minimum is therefore attained for the smallest possible value of t, i.e. for $t=3$.

4. Determine for which value(s) of a the linear system

$$\begin{aligned} 2x + y - z &= 0 \\ ax + y + 4z &= -\tfrac{5}{3} \\ 3x \qquad + az &= 1 \end{aligned}$$

has
(a) no solution,
(b) infinitely many solutions, and
(c) a unique solution.

We apply the Gaussian algorithm to the augmented matrix of the linear system

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ a & 1 & 4 & -\frac{5}{3} \\ 3 & 0 & a & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ a & 1 & 4 & -\frac{5}{3} \\ 3 & 0 & a & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1-\frac{a}{2} & 4+\frac{a}{2} & -\frac{5}{3} \\ 0 & -\frac{3}{2} & a+\frac{3}{2} & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3}(a+\frac{3}{2}) & -\frac{2}{3} \\ 0 & 1-\frac{a}{2} & 4+\frac{a}{2} & -\frac{5}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3}a-1 & -\frac{2}{3} \\ 0 & 0 & (*) & (**) \end{bmatrix} = B, \text{ where}$$

$$(*) = 4+\frac{a}{2} + \frac{2}{3}\left(1-\frac{a}{2}\right)\left(a+\frac{3}{2}\right) = 4+\frac{a}{2}+\frac{2}{3}\left(a+\frac{3}{2}-\frac{a^2}{2}-\frac{3a}{4}\right) = 4+1+a\left(\frac{1}{2}+\frac{2}{3}-\frac{1}{2}\right)-\frac{a^2}{3}$$
$$= \frac{1}{3}(15+2a-a^2) = -\frac{1}{3}(a-5)(a+3)$$

$$(**) = -\frac{5}{3}+\frac{2}{3}\left(1-\frac{a}{2}\right) = -\frac{5}{3}+\frac{2}{3}-\frac{a}{3} = -1-\frac{a}{3} = -\frac{1}{3}(a+3)$$

Hence

$$B = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3}a-1 & -\frac{2}{3} \\ 0 & 0 & -\frac{1}{3}(a-5)(a+3) & -\frac{1}{3}(a+3) \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3}a-1 & -\frac{2}{3} \\ 0 & 0 & (a-5)(a+3) & a+3 \end{bmatrix}$$

(a) no solution $\Leftrightarrow$ the last row is equivalent to $[0\ 0\ 0\ 1]$
$\Leftrightarrow (a-5)(a+3)=0$ but $a+3 \neq 0 \Leftrightarrow a=5$

(b) infinitely many solutions $\Leftrightarrow$ the last row is equivalent to
$[0\ 0\ 0\ 0] \Leftrightarrow (a-5)(a+3) = 0 = a+3 \Leftrightarrow a+3=0 \Leftrightarrow a=-3$

In this case, we have general solution is

$$B \sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & \frac{1}{3} \\ 0 & 1 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t+\frac{1}{3} \\ -t-\frac{2}{3} \\ t \end{bmatrix}, \ t \in \mathbb{R} \text{ arbitrary}$$

(c) unique solution: $(a-5)(a+3) \neq 0 \Leftrightarrow a \neq 5$ and $a \neq -3$.