1. (3 points) Let A be a 6×5-matrix of rank 4. Answer the following questions:

(a) If $AX = B$ is solvable, on how many parameters depends the general solution of $AX = B$?

The general solution depends on $n-r$ parameters, where in our case $n = 5 = \#$ of variables and $r = \text{rank}(A) = 4$. Thus the answer is $5-4=$

My answer: 1

(b) Let $U = \{B \in \mathbb{R}^6 : AX = B \text{ is solvable }\}$. What is the dimension of U?

We have $U = \text{col}(A)$. Hence $\dim U = \dim \text{col}(A) = \text{rank}(A) =$

My answer: 4

(c) What is the rank of A^T?

$\text{rank}(A) = \text{rank}(A^T)$, so $\text{rank}(A^T) =$

My answer: 4

2. (3 points) Find the inverse of

$$A = \begin{bmatrix} 2+i & 1 \\ 6+2i & 3 \end{bmatrix}$$

All entries of A^{-1} must be in the form $a + ib$ for $a, b \in \mathbb{R}$.

$\det(A) = 3\cdot(2+i) - 1(6+2i) = 6+3i-6-2i = i$, so $\det(A)^{-1} = \frac{1}{i} = -i$

It follows that

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} 3 & -1 \\ -(6+2i) & 2+i \end{bmatrix} = (-i)\begin{bmatrix} 3 & -1 \\ -(6+2i) & 2+i \end{bmatrix} = \begin{bmatrix} -3i & i \\ i(6+2i) & -i(2+i) \end{bmatrix}$$

$$= \begin{bmatrix} -3i & i \\ -2+6i & 1-2i \end{bmatrix}$$

3. (3 points) Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & -1 & 4 & 4 \\ 7 & 3 & -2 & 5 \\ 2 & 2 & 4 & 9 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ -1 & -1 & 4 & 4 \\ 7 & 3 & -2 & 5 \\ 2 & 2 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 6 & 7 \\ 0 & -4 & -16 & -16 \\ 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 6 & 7 \\ -4 & -16 & -16 \\ 0 & 0 & 3 \end{vmatrix} =$$

$$= 3 \begin{vmatrix} 0 & 6 \\ -4 & -16 \end{vmatrix} = 3 \cdot 4 \cdot 6 = 72$$

4. (3 points) The matrix $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ is diagonalizable:

$$A = PDP^{-1} \quad \text{for } P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

(you do not have to show this). Give a formula for x_k where (x_k) is the sequence determined by the recurrence relation $x_{k+2} = x_{k+1} + 2x_k$ for $k \geq 2$ and $x_0 = 3 = x_1$.

$\det(P) = -3$, $P^{-1} = \frac{-1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $P^{-1}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A^k V_0 = PD^k P^{-1}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2^{k+1} \\ (-1)^k \end{bmatrix}$

$= \begin{bmatrix} 2^{k+1} + (-1)^k \\ 2^{k+2} + (-1)^{k+1} \end{bmatrix} \Rightarrow x_k = 2^{k+1} + (-1)^k$

5. (3 points) For a vector space V of dimension 6 answer the following questions.

(a) Does every set of 7 vectors contain a spanning set of V?

No, only if it is a spanning set.

No

(b) Can every set of 5 linearly vectors in V be extended to a basis of V?

Yes

(c) How many subspaces of dimension 6 does V have?

Since V has dimension 6, any subspace of dimension 6 coincides with V. Hence:

only 1

6. (3 points) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 5x_2 \\ 7x_1 + 8x_2 \\ 3x_2 \end{bmatrix}.$$

standard matrix is $\begin{bmatrix} 4 & -5 \\ 7 & 8 \\ 0 & 3 \end{bmatrix}$

7. (3 points) The linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ has the property

$$T(x^2 - 2x + 1) = [3 \quad 0 \quad 4]^T, \quad T(3x + 4) = [-1 \quad 1 \quad -1]^T.$$

Find $T(x^2 - 8x + 7)$.

$$x^2 - 8x - 7 = s(x^2 - 2x + 1) + t(3x + 4) =$$
$$= sx^2 + (3t - 2s)x + (s + 4t)$$

$\Rightarrow \quad s = 1, \quad -8 = 3t - 2s, \quad -7 = s + 4t \quad \Rightarrow \quad s = 1, t = -2,$

hence $\quad x^2 - 8x + 7 = (x^2 - 2x + 1) - 2(3x + 4)$

Therefore $\quad T(x^2 - 8x + 7) = T(x^2 - 2x + 1) - 2T(3x + 4) =$

$$= \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3+2 \\ -2 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$$

8. (3 points) Check if the following subsets of $\mathbb{F}[\mathbb{R}]$ are linearly independent.

(a) $\{\cos x, \sin x\}$

Suppose $s(\cos x) + t(\sin x) = 0$.
For $x = 0$ we get $s \cdot 1 + t \cdot 0 = 0$, ie $s = 0$, so $t \sin x = 0$
For $x = \frac{\pi}{2}$ we get $t = 0$. Hence $\{\cos x, \sin x\}$ is lin indep.

(b) $\{\cos^2 x, \sin^2 1, 1\}$.

Not linearly independent, since

$$\cos^2 x + \sin^2 x - 1 = 0$$

is a non-trivial linear combination which gives 0.

9. (3 points) State 3 different statements which are equivalent to:

The columns of the $n \times n$ matrix A do not span $\mathbb{R}^n$.

conditions The condition is equivalent to "A is not invertible"

(I) Hence, any negation of the other 17 equivalent statements of the Invertible Matrix Theorem will do. Some examples:

(II)

- A is not invertible
- reduced row-echelon form of $A \neq I_n$
- $\mathrm{rank}(A) < n$ (or $\mathrm{rank}(A) \neq n$)
- there exists a column B such that $AX = B$ is not solvable

(III)

- A^T is not invertible
- $AX = 0$ has a non-trivial solution
- 0 is an eigenvalue of A
- the columns of A are linearly dependent
- the rows of A are lin. indep.
- the rows of A do not span $\mathbb{R}^n$

10. Consider the linear system

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 1 \\ -x & - & y & + & pz & = & 0 \\ 4x & + & 6y & & & = & 2p \end{array}$$

(a) **(3 points)** If C denotes the coefficient matrix of the system above and A its augmented matrix, find the rank of C and of A for all values of p and q.

(b) **(3 points)** Find all p so that the linear system above has

(i) a unique solution,
(ii) infinitely many solutions, and
(iii) no solution.

(c) **(2 points)** In case (ii) of (b) find all solutions.

(a) $A = [C \,|\, b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & -1 & p & 0 \\ 4 & 6 & 0 & 2p \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & p+1 & 1 \\ 0 & -2 & -4 & 2p-4 \end{bmatrix}$ (first three columns $= C$)

$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & p+1 & 1 \\ 0 & 0 & 2p-2 & 2p-2 \end{bmatrix} \Rightarrow \operatorname{rank}(A) = \begin{cases} 3 & p \neq 1 \\ 2 & p = 1 \end{cases} = \operatorname{rank} C$

(first three columns) = ref of C

(b) (i) unique sol $\iff p \neq 1$

(ii) infinitely many sol. $\iff p = 1$

(iii) this case does not occur

(c) For $p = 1$: $A \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding lin system
$x_1 = 3x_3 - 1$
$x_2 = -2x_3 + 1$
x_3 free

general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t-1 \\ -2t+1 \\ t \end{bmatrix}$, t free parameter

11. (7 points) For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 3 & 6 & 10 & 23 \\ 4 & 8 & 10 & 24 \end{bmatrix}$$

find

(a) **(2 points)** the reduced row echelon form,
(b) **(1 point)** a basis of the row space,
(c) **(1 point)** a basis of the column space,
(d) **(2 points)** a basis of the null space,
(e) **(1 point)** the dimension of $\mathrm{Col}\,(A)^{\perp}$, the orthogonal complement of the column space of A

(a) $\begin{bmatrix} 1 & 2 & 3 & 7 \\ 3 & 6 & 10 & 23 \\ 4 & 8 & 10 & 24 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) A basis of the row space of A is $\{[1\ 2\ 0\ 1], [0\ 0\ 1\ 2]\}$

(c) A basis of the column space of A is $\left\{\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \\ 10 \end{bmatrix}\right\}$

(d) Since $A \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ an equivalent linear system is

$x_1 + 2x_2 + x_4 = 0$ free variables are $x_2 = s$, $x_4 = t$
$x_3 + 2x_4 = 0$

general solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$\Rightarrow$ a basis of the null space of A is $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$

(e) Since $\mathrm{col}(A) \subset \mathbb{R}^3$ and $\dim \mathrm{col}(A) = 2$, we have

$\dim \mathrm{col}(A)^{\perp} = 3 - \dim \mathrm{col}(A) = 3 - 2 = 1$

(see Thm. 3 in §4.6)

12. (a) **(5 points)** Find all eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 3 \\ -2 & 0 & -1 \end{bmatrix}.$$

(b) Is A diagonalizable?

(c) Is A invertible?

(a) characteristic polynomial is

$$c_A(x) = \begin{vmatrix} x-2 & 0 & -1 \\ 2 & x+1 & -3 \\ 2 & 0 & x+1 \end{vmatrix} = (x+1)\begin{vmatrix} x-2 & -1 \\ 2 & x+1 \end{vmatrix} =$$

$$= (x+1)\big((x-2)(x+1)+2\big) = (x+1)\left(x^2-x-2+2\right) = (x+1)(x^2-x)$$

$$= (x+1)x(x-1)$$

Hence the eigenvalues of A are ± 1 and 0

(b) Yes, since all its eigenvalues have multiplicity 1.

(c) No, since 0 is an eigenvalue

13. The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$$

are ± 3. (You do not need to show this.)

(a) **(4 points)** For each eigenvalue of A find a basis of the corresponding eigenspace.

(b) **(2 points)** Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Justify your answer.

(a) $\lambda = 3$: $3I_3 - A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

corresponding system is $x_1 = x_2 - x_3$, general solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

basic eigenvectors: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -3$: $-3I_3 - A = \begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 2 & -8 & 2 \\ -4 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -6 & 2 \\ 0 & -6 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$ corresp. lin. syst $x_1 = 1/3\, x_3$, $x_2 = 1/3\, x_3$

general solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, basic e-vector $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

(b) Yes, A is diagonalizable since for each e-value, $\dim E_\lambda(A) =$ multiplicity of λ. Equivalently, we have a basis of e-vectors.

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

14. Let $U = \{[x \quad y \quad z]^T \in \mathbb{R}^3 : x + 2y + 3z = 0\}$.

(a) **(3 points)** Show that U is a subspace of $\mathbb{R}^3$ and find a basis of U.

(b) **(3 points)** Find an orthogonal basis of U.

(c) **(2 points)** Find the orthogonal projection of $[0 \quad 1 \quad 0]^T$ onto U.

(a) We have $x+2y+3z=0 \Leftrightarrow x=-2y-3z$. Hence

$$U = \left\{ \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} : y,z \in \mathbb{R} \right\} = \left\{ y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : y,z \in \mathbb{R} \right\} =$$

$$= \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$ As a span, U is a subspace of $\mathbb{R}^3$.

The spanning set $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is lin. indep since

$$y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y = 0 = z.$$

Therefore $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of U.

(b) We apply Gram-Schmidt to the basis of (a). $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$$F_1 = X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad F_2 = X_2 - \frac{X_2 \cdot X_1}{X_1 \cdot X_1} X_1 = X_2 - \frac{6}{5} X_1 = \begin{bmatrix} -3 + \frac{6}{5}\cdot 2 \\ 0 - 6/5 \\ 1 + 0 \end{bmatrix} = \begin{bmatrix} -3 + \frac{12}{5} \\ -6/5 \\ 1 \end{bmatrix}$$

$$\Rightarrow F_2 = \frac{1}{5}\begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}.$$

$\Rightarrow \{F_1, F_2\}$ is an orthogonal basis of U

(c) $$\operatorname{proj}_U\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \frac{[0\ 1\ 0]^T \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{[0\ 1\ 0]^T \cdot F_2}{F_2 \cdot F_2} F_2 = \frac{1}{5} F_1 - \frac{\frac{6}{5}}{\frac{9+36+25}{25}} F_2$$

$$= \frac{1}{5} F_1 - \frac{6}{\cancel{5}} \cdot \frac{\cancel{25}^{5}}{70} F_2 = \frac{1}{5} F_1 - \frac{3}{7} F_2 = \frac{1}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{35}\begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} -14 + 9 \\ 7 + 18 \\ -15 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} -5 \\ 25 \\ -15 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}.$$

15. Recall that A^T denotes the transpose of an $n \times n$-matrix.

(a) **(3 points)** Show that $U = \{A \in \mathbb{M}_{nn} : A = A^T\}$ is a subspace of $\mathbb{M}_{nn}$. You can get 2 extra points if you give **two different** proofs for this.

(b) **(4 points)** For $n = 2$ find a basis of U.

(a) 1^{st} method: We check the 3 conditions of the subspace test:

(i) $0 \in U$: Here 0 = zero matrix, and since its transpose is again 0, it lies in U.

(ii) Suppose $A, B \in U$, i.e. $A^T = A$ and $B^T = B$. Since then $(A+B)^T = A^T + B^T = A + B$, we see that $A+B \in U$.

(iii) Suppose $A \in U$, i.e. $A^T = A$, and let $s \in \mathbb{R}$. Then $(sA)^T = sA^T = sA$, whence $sA \in U$.

2^{nd} method: The map $T: \mathbb{M}_{nn} \to \mathbb{M}_{nn}$, $T(A) = A - A^T$ is linear. Indeed, for $A, B \in \mathbb{M}_{nn}$ we have $T(A+B) = A+B - (A+B)^T = A + B - A^T - B^T = T(A) + T(B)$, and for $s \in \mathbb{R}$, $A \in \mathbb{M}_{nn}$ we have $T(sA) = sA - (sA)^T = s(A - A^T) = sT(A)$. The set $U = \{A \in \mathbb{M}_{nn} : A - A^T = 0\} = \{A \in \mathbb{M}_{nn} : T(A) = 0\} = \text{Ker}(T)$, hence U is a subspace by § 5.3.2, lemma 2.

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{22}$. Then $A = A^T \Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Leftrightarrow b = c$.

Hence $U = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\} = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$
$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. We check that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is l.i.:

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow a = b = d = 0$$

Hence $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of U.

16. **(4 bonus points)** (a) Let V be a vector space, and let $v_1, \ldots, v_n \in V$. Give the definition of Span $\{v_1, \ldots, v_n\}$.

By definition, the span consists of all linear combinations of V. That is,

$$\text{span}\{v_1, \ldots, v_n\} = \{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n : a_i \in \mathbb{R}\}$$

(b) Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces. Use the formula

$$\dim ker(T) + \dim im(T) = \dim V$$

to show that if T is injective (= one-to-one) then $\dim V \leq \dim W$.

We have seen that T is injective $\Leftrightarrow$ $\ker(T) = \{0\}$ (Th 4 in §5.3). Hence if T is injective, then $\dim \ker(T) = 0$. Therefore $\dim im(T) = \dim V$. But $im(T)$ is a subspace of W (Lemma 2 in §5.3), therefore $\dim im(T) \leq \dim W$ (Th 5 in §5.2). Altogether we get $\dim V = \dim im(T) \leq \dim W$.