# University of Ottawa

# Department of Mathematics and Statistics

## MAT 1341: Introduction to Linear Algebra

**Instructor: Erhard Neher**

**Final Exam; July 21, 2008**

**Family Name:** ______________________________

**First Name:** ______________________________

**Student number:** ______________________________

### Please read these instructions carefully:

- You have 180 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.
- If you do not have enough space, use the back of the pages and clearly indicate this. There is an extra page at the end of the exam.
- Where it is possible to check your work, do so.
- Read each question carefully - you will save yourself time and unnecessary grief later on.

**Good luck! Bonne chance!**

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| Quest. | 1 – 9 | 10 | 11 | 12 – 13 | 14 | 15 | 16 | Total |
|---|---|---|---|---|---|---|---|---|
| maximal | 27 | 8 | 7 | 11 | 8 | 7 | (4 bonus) | 68 |
| score | | | | | | | | |

**1. (3 points)** Let $A$ be a $6 \times 5$-matrix of rank 4. Answer the following questions:

(a) If $AX = B$ is solvable, on how many parameters depends the general solution of $AX = B$?

**My answer:**________________

(b) Let $U = \{B \in \mathbb{R}^6 : AX = B \text{ is solvable } \}$. What is the dimension of $U$?

**My answer:**________________

(c) What is the rank of $A^T$?

**My answer:**________________

**2. (3 points)** Find the inverse of

$$A = \begin{bmatrix} 2+i & 1 \\ 6+2i & 3 \end{bmatrix}$$

All entries of $A^{-1}$ must be in the form $a + ib$ for $a, b \in \mathbb{R}$.

**My answer:**________________________________

**3. (3 points)** Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & -1 & 4 & 4 \\ 7 & 3 & -2 & 5 \\ 2 & 2 & 4 & 9 \end{bmatrix}$$

**My answer:**________________

**4. (3 points)** The matrix $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ is diagonalizable:

$$A = PDP^{-1} \quad \text{for } P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

(you do not have to show this). Give a formula for $x_k$ where $(x_k)$ is the sequence determined by the recurrence relation $x_{k+2} = x_{k+1} + 2x_k$ for $k \geq 2$ and $x_0 = 3 = x_1$.

**My answer:**________________

**5. (3 points)** For a vector space $V$ of dimension 6 answer the following questions.

(a) Does every set of 7 vectors contain a spanning set of $V$?

**My answer:**________________

(b) Can every set of 5 linearly independent vectors in $V$ be extended to a basis of $V$?

**My answer:**________________

(c) How many subspaces of dimension 6 does $V$ have?

**My answer:**________________

**6. (3 points)** Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 5x_2 \\ 7x_1 + 8x_2 \\ 3x_2 \end{bmatrix}.$$

**My answer:**______________________________

**7. (3 points)** The linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ has the property

$$T(x^2 - 2x + 1) = [3 \quad 0 \quad 4]^T, \quad T(3x + 4) = [-1 \quad 1 \quad -1]^T.$$

Find $T(x^2 - 8x - 7)$.

**My answer:**______________________________

**8. (3 points)** Check if the following subsets of $\mathbb{F}[\mathbb{R}]$ are linearly independent.

(a) $\{\cos x, \sin x\}$

**My answer:**________________

(b) $\{\cos^2 x, \sin^2 x, 1\}$.

**My answer:**________________

**9. (3 points)** State 3 different statements which are equivalent to:

The columns of the $n \times n$ matrix $A$ do not span $\mathbb{R}^n$.

(I)

(II)

(III)

**10.** Consider the linear system

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ -x & - & y & + & pz & = & 0 \\ 4x & + & 6y & & & = & 2p \end{array}$$

(a) **(3 points)** If $C$ denotes the coefficient matrix of the system above and $A$ its augmented matrix, find the rank of $C$ and of $A$ for all values of $p$.

(b) **(3 points)** Find all $p$ so that the linear system above has

(i) a unique solution,
(ii) infinitely many solutions, and
(iii) no solution.

(c) **(2 points)** In case (ii) of (b) find all solutions.

**11. (7 points)** For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 3 & 6 & 10 & 23 \\ 4 & 8 & 10 & 24 \end{bmatrix}$$

find

(a) **(2 points)** the reduced row echelon form,
(b) **(1 point)** a basis of the row space,
(c) **(1 point)** a basis of the column space,
(d) **(2 points)** a basis of the null space,
(e) **(1 point)** the dimension of $col\,(A)^{\perp}$, the orthogonal complement of the column space of $A$.

**12.** (a) **(3 points)** Find all eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 3 \\ -2 & 0 & -1 \end{bmatrix}.$$

(b) **(1 point)** Is $A$ diagonalizable?

(c) **(1 point)** Is $A$ invertible?

**13.** The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$$

are $\pm 3$. (You do not need to show this.)

(a) **(4 points)** For each eigenvalue of $A$ find a basis of the corresponding eigenspace.

(b) **(2 points)** Decide if $A$ is diagonalizable or not. If yes, give an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$. Justify your answer.

**14.** Let $U = \{[x \quad y \quad z]^T \in \mathbb{R}^3 : x + 2y + 3z = 0\}$.

(a) **(3 points)** Show that $U$ is a subspace of $\mathbb{R}^3$ and find a basis of $U$.

(b) **(3 points)** Find an orthogonal basis of $U$.

(c) **(2 points)** Find the orthogonal projection of $[0 \quad 1 \quad 0]^T$ onto $U$.

**15.** Recall that $A^T$ denotes the transpose of an $n \times n$-matrix.

(a) **(3 points)** Show that $U = \{A \in \mathbb{M}_{nn} : A = A^T\}$ is a subspace of $\mathbb{M}_{nn}$. You can get 2 **extra points** if you give **two different** proofs for this.

(b) **(4 points)** For $n = 2$ find a basis of $U$.

**16.** (**4 bonus points**) (a) Let $V$ be a vector space, and let $v_1, \ldots, v_n \in V$. Give the definition of $span\,\{v_1, \ldots, v_n\}$.

(b) Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces. Use the formula

$$\dim ker(T) + \dim im(T) = \dim V$$

to show that if $T$ is injective (= one-to-one) then $\dim V \leq \dim W$.

(extra page)