1. The volume of the parallelepiped determined by $\vec{u} = [1 \quad 1 \quad 2]^T$, $\vec{v} = [1 \quad -1 \quad -1]^T$ and $\vec{w} = [0 \quad 2 \quad 4]^T$ is

A. 7
B. $\frac{5}{2}$
C. -4
D. $\frac{3}{2}$
E. 5
F. 2

(see §3.5, Theorem 6)

$$\left|\vec{u}, \vec{v}, \vec{w}\right| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & -1 & 4 \end{vmatrix} =$$

My answer: F

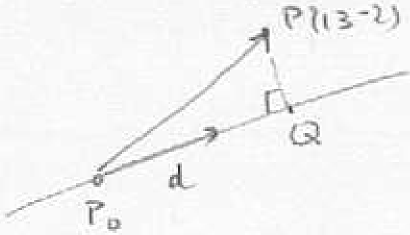
$$= \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = -4+2-(4-4) = -2$$

$\Rightarrow$ volume is $|\det(\vec{u}, \vec{v}, \vec{w})| = 2$

2. The point on the line through $P_0(2, 0, -1)$ with direction vector $\vec{d} = [1 \quad 1 \quad 0]^T$ which is closed to $P(1, 3, -2)$ is

A. $[3 \quad 1 \quad -1]^T$
B. $[2 \quad 0 \quad -1]^T$
C. $[1 \quad -1 \quad 1]^T$
D. $[4 \quad 2 \quad -1]^T$
E. $[5 \quad 3 \quad -1]^T$
F. $[5 \quad 3 \quad -1]^T$

(see §3.2, Example 9)



My answer: A

$$\overrightarrow{P_0P} = [1, 3, -2]^T - [2, 0, -1]^T = [-1, 3, -1]^T$$

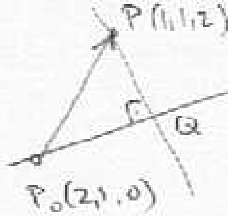
$$\mathrm{proj}_{\vec{d}}(\overrightarrow{P_0P}) = \frac{\overrightarrow{P_0P} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{-1+3+0}{1+1} \vec{d} = \vec{d} = [1, 1, 0]^T$$

Q = closest to P is $\overrightarrow{OP_0} + \mathrm{proj}_{\vec{d}}(\overrightarrow{P_0P}) = [2, 0, -1]^T + [1, 1, 0]^T =$

$= [3, 1, -1]^T$

3. The direction vector of the line through $P(1,1,2)$, intersecting the line $[x \ \ y \ \ z]^T = [2 \ \ 1 \ \ 0]^T + t[1 \ \ 1 \ \ 1]^T$ and perpendicular to it, is

A. $[0 \ \ 1 \ \ -1]^T$
B. $[2 \ \ 1 \ \ -3]^T$
C. $[1 \ \ -2 \ \ 1]^T$
D. $[4 \ \ 1 \ \ -5]^T$
E. $[3 \ \ -1 \ \ 2]^T$
F. such a line does not exist.



(see DGD, Monday, May 5)

My answer: D

Let P_0 be a point on the line, say $[2,1,0]$. Hence $\overrightarrow{P_0P} = [1,1,2]^T - [2,1,0]^T = [-1,0,2]^T$. We know that $\vec{d} = [1,1,1]^T$ is the direction vector of the line. Hence $\overrightarrow{P_0Q} = \operatorname{proj}_{\vec d}(\overrightarrow{P_0P}) = \frac{[-1,0,2]^T \cdot [1,1,1]^T}{1+1+1}\vec{d} = \frac{-1+2}{3}\vec{d} = \frac{1}{3}[1,1,1]^T$

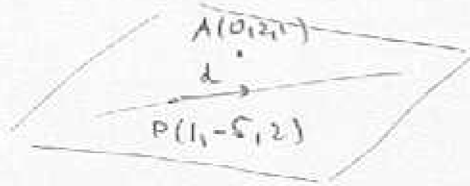
If Q is the point on line L closest to P, we get

$\overrightarrow{OQ} = \overrightarrow{OP_0} + \operatorname{proj}_{\vec d}(\overrightarrow{P_0P}) = [2,1,0]^T + \frac{1}{3}[1,1,1]^T = [\frac{7}{3}, \frac{4}{3}, \frac{1}{3}]^T = \frac{1}{3}[7,4,1]^T$

The point Q lies on the line we are looking for. Hence, a direction vector for this line is $\overrightarrow{QP} = [1,1,2]^T - \frac{1}{3}[7,4,1]^T = [-\frac{4}{3}, -\frac{1}{3}, \frac{5}{3}]^T = \frac{1}{3}[-4,-1,5]^T$. But direction vectors can be replaced by a non-zero scalar multiple, so $-3\left(\frac{1}{3}[-4,-1,5]^T\right) = [4,1,-5]^T$ is a direction vector

4. The equation of the plane containing the point $A(0,-2,1)$ and the line $[x \ \ y \ \ z]^T = [1 \ \ -5 \ \ 2]^T + t[-1 \ \ 1 \ \ 5]^T$ is

A. $7x - 2y + z = 5$
B. $8x + 3y + z = -5$
C. $2x + y + z = -1$
D. $-x + z = 1$
E. $5x + 2y + z = -3$
F. $-4x - y + z = 3$



My answer: B

The point $P(1,-5,2)$ lies on the plane.

Hence $\overrightarrow{PA} = [0,-2,1]^T - [1,-5,2]^T = [-1,3,-1]^T$ and the direction vector $\vec{d} = [-1,1,5]^T$ lie in the plane. A normal is therefore given by

$$\vec{n} = \overrightarrow{PA} \times \vec{d} = \begin{vmatrix} \vec i & \vec j & \vec k \\ -1 & 3 & -1 \\ -1 & 1 & 5 \end{vmatrix} = \left[\begin{vmatrix} 3 & -1 \\ 1 & 5 \end{vmatrix}, -\begin{vmatrix} -1 & -1 \\ -1 & 5 \end{vmatrix}, \begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} \right] = [16, 6, 2]$$

Hence an eq of the plane is $16x + 6y + 2z = k$. We determine k by evaluating the equation for $A(0,-2,1)$: $k = 16\cdot 0 + 6(-2) + 2\cdot(1) = -10$

Hence $16x + 6y + 2z = -10$ is an equation for the plane.

Equivalently: $8x + 3y + z = -5$

5. The vector equation of the line through $[3\ \ 1\ \ 5]^T$ and $[4\ \ 2\ \ 1]^T$ is

A. $[3\ \ 1\ \ 5]^T + t[1\ \ -5\ \ 2]^T$
B. $[4\ \ 2\ \ 1]^T + t[1\ \ 1\ \ -3]^T$
C. $[3\ \ 1\ \ 5]^T + t[-1\ \ -1\ \ 4]^T$
D. $[4\ \ 2\ \ 1]^T + t[1\ \ -1\ \ 3]^T$
E. $[3\ \ 1\ \ 5]^T + t[-2\ \ 3\ \ -1]^T$
F. $[4\ \ 2\ \ 1]^T + t[1\ \ -5\ \ 3]^T$

A direction vector is

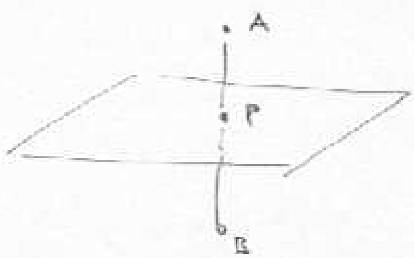
$\vec{d} = [3,1,5] - [4,2,1] = [-1,-1,4]$

Hence a vector equation is C

My answer: C

6. The equation of the plane, each point of which has the same distance from the points $A(2,4,-2)$ and $B(3,2,-1)$, is

A. $2x + 4y - 2z = 16$
B. $7x - 4y + 3z = 9$
C. $x - 2y + z = -5$
D. $x + 3y - 4z = 0$
E. $3x + 2y - z = 14$
F. $3x - 4y - z = 9$



My answer: C

$\vec{AB} = [3,2,-1]^T - [2,4,-2]^T = [1,-2,1]^T$ is a normal of the plane.
Hence an equation of the plane has the form
$x - 2y + z = k$, for some constant k. The point $\vec{OP} = \vec{OA} + \frac{1}{2}\vec{AB}$
$= [2,4,-2]^T + \frac{1}{2}[1,-2,1]^T = [\frac{5}{2}, 3, -\frac{3}{2}]^T$ lies on the plane. Therefore
$k = \frac{5}{2} - 2(3) + (-\frac{3}{2}) = \frac{5}{2} - 6 - \frac{3}{2} = -5$.

7. Find the correct combination of true/false for the following three statements.

- If $\vec{u}$ and $\vec{v}$ are orthogonal, then $-3\vec{u}$ and $2\vec{v}$ are also orthogonal.
- If the projection vector $\text{proj}_{\vec{d}}(\vec{v}) = 0$, then $\vec{v} = 0$.
- If $\vec{v} \cdot \vec{w} = 0$, then $\vec{v} = 0$ or $\vec{w} = 0$.

A. true, true, true
B. true, true, false
C. false, true, false
D. false, false, true
E. false, false, false
F. true, false, false

(1) true since $(-3\vec{u}) \cdot (2\vec{v}) = (-3)(2)\,\vec{u} \cdot \vec{v} = -6\,\vec{u} \cdot \vec{v} = 0$

(2) false: $\text{proj}_{\vec{d}}(\vec{v}) = 0 \Leftrightarrow \vec{v} \cdot \vec{d} = 0$, i.e. $\vec{v} \perp \vec{d}$

(3) false: $\vec{v} \cdot \vec{w} = 0$ only means $\vec{v} \perp \vec{w}$

My answer: F

8. Find the correct combination of true/false for the following three statements.

- The equation $3z^3 - 4z^2 + 7z - 9 = 0$ does not have a solution in the complex numbers $\mathbb{C}$.
- The complex conjugate $\bar{z}$ of a complex number z is always different from z.
- For any complex number z we have $z\bar{z} > 0$.

A. true, true, true
B. true, true, false
C. false, true, false
D. false, false, true
E. false, false, false
F. true, false, false

(1) false (Fundamental Theorem of Algebra)

(2) false: $\bar{z} = z \Leftrightarrow z$ is a real number

(3) false: $z\bar{z} \geq 0$ only. $z\bar{z} = 0 \Leftrightarrow z = 0$.

My answer: E

9. The complex number z satisfying $\overset{3}{\cancel{2}}z + 1 + 2i = 4iz + 3 + i$ is

A. $\frac{1}{10}(1+7i)$

B. $\frac{1}{5}(2+i)$

C. $\frac{1}{5}(2+3i)$

D. $\frac{1}{10}(1-i)$

E. $\frac{1}{5}(1+2i)$

F. $\frac{1}{10}(2-3i)$

My answer: B

$$3z + 1 + 2i = 4iz + 3 + i \iff$$

$$(3-4i)z = 3 + i - 1 - 2i = 2 - i \iff$$

$$z = \frac{2-i}{3-4i} = \frac{(2-i)(3+4i)}{(3-4i)(3+4i)} = \frac{6-4i^2-3i+8i}{9+16} = \frac{6+4+5i}{25}$$

$$= \frac{10+5i}{25} = \frac{1}{5}[2+i]$$