1. (3 points) Write $X = [1 \ \ 3 \ \ 1 \ \ 0]^T$ as a linear combination of the vectors $[2 \ \ 4 \ \ 0 \ \ -6]^T$ and $[0 \ \ 3 \ \ 3 \ \ 9]^T$.

Reference. Similar to §4.1 #6c, done in the DGD on June 16. Also, #6bd are suggested exercises.

We need to find $x, y \in \mathbb{R}$ such that

$$x[2\ 4\ 0\ -6]^T + y[0\ 3\ 3\ 9]^T = [1\ 3\ 1\ 0]^T$$

This is equivalent to the linear system

$$\left.\begin{array}{r} 2x \qquad = 1 \\ 4x + 3y = 3 \\ 3y = 1 \\ -6x + 3y = 0 \end{array}\right\}$$

From the 1st equation $x = \frac{1}{2}$, and from the 3rd equation $y = \frac{1}{3}$. One checks that $x = \frac{1}{2}$, $y = \frac{1}{3}$ also solve the other two equations. Hence

My answer: $\frac{1}{2}\begin{bmatrix} 2 \\ 4 \\ 0 \\ -6 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

2. Let U be a subspace of $\mathbb{R}^{10}$ which is spanned by 6 vectors. Are the following claims true or false? Answer with T for "true" or F for "false".

(a) (1 point) $\dim U = 6$.

Only $\dim U \leq 6$ follows. Note the spanning set need not be a basis.

My answer: F

(b) (1 point) Any set of 7 vectors in U is linearly dependent.

Fundamental Theorem in §4.3

My answer: T

(c) (1 point) Every spanning set of U has at most 6 vectors.

One can always add vectors to a spanning set and get a new, bigger spanning set.

My answer: F

Reference. Theorems of section §4.3

3. Let A be a 3×7 matrix. Are the following claims true or false? Answer with T for "true" or F for "false".

(a) (1 point) If the rows of A are linearly independent, then $rowA = \mathbb{R}^4$.

The rows of A are vectors in $\mathbb{R}^7$, and not in $\mathbb{R}^4$

My answer: F

(b) (1 point) The dimension of the null space of A is at least 4.

Let r be the rank of A. Then $r \le 3$. By Theorem 6 in §4.4, $\dim \text{null } A = n - r =$
$= 7 - r \geq 4$.

My answer: T

(c) (1 point) If A has a row of zeros, then rank$(A) \leq 2$

My answer: T

Reference. (a) §4.4, #7b (suggested exercise); (b) §4.4, #11 (done in the DGD on June 16); (c) §4.4, #7a (done in the DGD on June 16)

4. Are the following subsets U subspaces of the indicated vector spaces V? Answer with Y for "yes" or N for "no".

(a) (1 point) $U = \{xp(x) : p \in \mathbb{P}_3\}$ in $V = \mathbb{P}$.

(done in class on June 23)

My answer: Y

(b) (1 point) $U = \{A \in \mathbb{M}_{2,2} : \det(A) = 1\}$ in $V = \mathbb{M}_{2,2}$.

U does not contain the zero matrix, whose determinant is 0

My answer: N

(c) (1 point) $U = \{f \in \mathbb{F}[0,1] : f(\frac{1}{2}) = 0\}$ in $V = \mathbb{F}[0,1]$.

all conditions of the subspace test are fulfilled, see the DGD

My answer: Y

Reference. (a) done in class on June 23; (b) similar to exercise 7c, done in the DGD on June 23; (c) similar to suggested exercise 8b in §5.1 and also to 8c done in the DGD on June 23.

5. (5 points) Show that

$$U = \{[a \ \ b \ \ c]^T \in \mathbb{R}^3 : a - b + c = 0\}$$

is a subspace of $\mathbb{R}^3$, find a basis of U and calculate its dimension.

Reference. Similar to §4.3 #7c, done in the DGD on June 16; also #6b is a suggested exercise.

Since $a - b + c = 0 \Leftrightarrow b = a + c$, we get

(2) $U = \{[a, a+c, c]^T \in \mathbb{R}^3 : a, c \in \mathbb{R}\} =$

$= \{a[1,1,0]^T + c[0,1,1]^T : a, c \in \mathbb{R}\}$

$= \text{span}\{[1,1,0]^T, [0,1,1]^T\}$

As a span, U is a subspace. Since we have a spanning set, we check if it is linearly independent:

(2) Suppose $a[1,1,0]^T + c[0,1,1]^T = [0,0,0]^T$. Then

$[a, a+c, c]^T = [0,0,0]^T$

Hence $a = c = 0$. This shows that $\{[1,1,0]^T, [0,1,1]^T\}$ is a basis.

My answer for basis: $\{[1,1,0]^T, [0,1,1]^T\}$

(1) My answer for the dimension: 2

6. (5 points) Find a basis of the row space of A and of the column space of A, and determine the rank of A where A is the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 4 & 8 & -3 & 10 & 4 \\ 2 & 4 & 1 & 3 & 15 \\ -1 & -2 & 0 & -1 & -4 \end{bmatrix}$$

Reference. Same as question 1 of assignment 3.

We find a row-echelon form of A:

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 4 & 8 & -3 & 10 & 4 \\ 2 & 4 & 1 & 3 & 15 \\ -1 & -2 & 0 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 3 & -3 & 15 \\ 0 & 0 & -1 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can now get the answers to the questions from the Great Rank Theorem

My answer for a basis of the row space of A: $\{ [1, 2, -1, 3, 0], [0, 0, 1, -2, 4], [0, 0, 0, 1, 1] \}$

My answer for a basis of the column space of A: $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \\ 3 \\ -1 \end{bmatrix} \right\}$

My answer for the rank of A: 3

7. (a) **(3 points)** Apply the Gram-Schmidt algorithm to convert

$$X_1 = [1\ \ 1\ \ 1\ \ 1]^T,\quad X_2 = [3\ \ 1\ \ 3\ \ 1]^T,\quad X_3 = [1\ \ 3\ \ -1\ \ 1]^T$$

into an orthogonal basis of $U = \operatorname{Span}\{X_1, X_2, X_3\}$. (Continued on next page.)

Reference. Assignment 3, question 2; a similar question as done in the DGD on June 23.

We put $F_1 = X_1$ and calculate

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{F_1 \cdot F_1} F_1 = [3,1,3,1]^T - \frac{8}{4}[1,1,1,1]^T =$$

$$= [3,1,3,1]^T - 2[1,1,1,1]^T = [1,-1,1,-1]^T.$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{F_1 \cdot F_1} F_1 - \frac{X_3 \cdot F_2}{F_2 \cdot F_2} F_2 =$$

$$= X_3 - \frac{[1,3,-1,1] \cdot [1,1,1,1]}{4} F_1 - \frac{[1,3,-1,1] \cdot [1,-1,1,-1]}{4} F_2$$

$$= X_3 - \frac{4}{4} F_1 - \frac{1-3-1-1}{4} F_2 = X_3 - F_1 + F_2$$

$$= [1,3,-1,1]^T - [1,1,1,1]^T + [1,-1,1,-1]^T$$

$$= [1,1,-1,-1]^T.$$

(b) **(2 points)** The vectors $F_1 = [1 \quad -2 \quad 1]^T$ and $F_2 = [0 \quad 1 \quad 2]^T$ are orthogonal. Find the orthogonal projection of the vector $X = [11 \quad 4 \quad 3]^T$ onto the subspace $U = \operatorname{Span}\{F_1, F_2\}$.

Reference. Assignment 3, problem 3. A similar problem (exercise 1c of §4.6) was done in the DGD on June 23.

$$\mathrm{proj}_U(X) = \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{X \cdot F_2}{F_2 \cdot F_2} F_2 =$$

$$= \frac{[11,4,3]\cdot[1,-2,1]}{6} F_1 + \frac{[11,4,3]\cdot[0,1,2]}{5} F_2$$

$$= \frac{11-8+3}{6} F_1 + \frac{4+6}{5} F_2 = F_1 + 2F_2 =$$

$$= [1,-2,1]^T + [0,2,4]^T = [1,0,5]^T.$$

8. (**2 bonus points**) Let $S = \{X_1, X_2, \dots, X_k\}$ be an orthogonal subset of $\mathbb{R}^n$. Prove that S is linearly independent.

Suppose $s_1X_1 + s_2X_2 + \dots + s_kX_k = 0$. Fix i with $1 \le i \le k$, and take the dot product with X_i:

$$0 = 0 \cdot X_i = (s_1X_1 + s_2X_2 + \dots + s_kX_k) \cdot X_i =$$

$$= s_1 \underbrace{X_1 \cdot X_i}_{=0} + \dots + s_{i-1} \underbrace{X_{i-1} \cdot X_i}_{=0} + s_i X_i \cdot X_i + s_{i+1} \underbrace{X_{i+1} \cdot X_i}_{=0} + \dots + s_k \underbrace{X_k \cdot X_i}_{=0}$$

$$= s_i X_i \cdot X_i ,$$

when we used that $X_j \cdot X_i = 0$ for $j \neq i$. Since $X_i \cdot X_i > 0$ by definition of an orthogonal subset of $\mathbb{R}^n$, the equation $0 = s_i (X_i \cdot X_i)$ forces $s_i = 0$. Thus S is lin. independent.