1. (1 point) If $\det(A) = -\frac{1}{2}$, $\det(B) = 4$ and $\det(C) = 5$ then $\det(A^4 B^2 C^T A^{-1})$ is

A. -20.
B. $-\frac{2}{5}$.
C. -10.
D. 20.
E. $-\frac{5}{2}$.
F. 4.

$$\det(A^4 B^2 C^T A^{-1}) = (\det A)^4 (\det B)^2 \det C \frac{1}{\det A} =$$
$$= (\det A)^3 (\det B)^2 \det C = -\frac{1}{8} \cdot 16 \cdot 5 = -2 \cdot 5 = -10$$

My answer: C

2. (1 point) The eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ are

A. 4 and -1.
B. this matrix does not have eigenvalues, even complex ones.
C. $\pm\sqrt{3}$.
D. only 1 is an eigenvalue.
E. $1 \pm i$.
F. $2 \pm i\sqrt{3}$.

$$c_A(\lambda) = \begin{vmatrix} \lambda-1 & -1 \\ 1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 + 1 \quad \text{so}$$

$$c_A(\lambda) = 0 \Leftrightarrow (\lambda-1)^2 = - \Leftrightarrow \lambda - = \pm i \Leftrightarrow \lambda = 1 \pm i$$

My answer: E

3. (1 point) Find all values of x for which the vectors (1, 2, 2), (0, 1, $3-x$) and (1, x, 0) are linearly independent.

A. $x \neq -2$ and $x \neq -6$.
B. $x \neq 1$ and $x \neq 4$.
C. $x \neq 0$ and $x \neq -3$.
D. $x = 2$ and $x = -4$.
E. $x \neq 3$ and $x \neq 5$.
F. $x \neq 2$ and $x \neq 5$.

The vectors are li $\Leftrightarrow$ $\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3-x \\ 1 & x & 0 \end{vmatrix} \neq 0$

We have

$$\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3-x \\ 1 & x & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3-x \\ 0 & x-2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3-x \\ 0 & 0 & a \end{vmatrix}, \text{ where}$$

$$a = -2 - (x-2)(3-x) = -2 + (x-2)(x-3) = -2 + x^2 - 5x + 6 = x^2 - 5x + 4 = (x-4)(x-1).$$

Since $\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3-x \\ 1 & x & 0 \end{vmatrix} = a$, the vectors are li $\Leftrightarrow a \neq 0 \Leftrightarrow x-4 \neq 0$ and $x-1 \neq 0 \Leftrightarrow x \neq 4$ and $x \neq$

My answer: B

4. (1 point) If the vector (0, −5, 4, −9) is written as a linear combination of the vectors

(1, 0, 0, 1), (0, 1, 4, 1) and(1, 3, 0, 6)

then the sum of the coefficients in this linear combination is

A. 1.
B. 2.
C. 3.
D. 0.
E. 4.
F. −2.

We have to find x, y, z s.th.

$$x(1\ 0\ 0\ 1) + y(0\ 1\ 4\ 1) + z(1\ 3\ 0\ 6) = (0, -5, 4, -9)$$

Thus we need to solve the linear system

$$\begin{cases} x + z = 0 \\ y + 3z = -5 \\ 4y = 4 \\ x + y + 6z = -9 \end{cases} \Leftrightarrow \begin{cases} y = 1 \\ x + z = 0 \\ 3z = -6 \\ x + 6z = -10 \end{cases} \Leftrightarrow \begin{cases} y = 1 \\ z = -2 \\ x = 2 \end{cases}$$

Thus $x + y + z =$

My answer: A

5. (1 point) Let U be a subspace of $\mathbb{R}^n$ with $\dim U = p$. Which of the following statements are true? Give the correct combination of answers.

- Any set of $p+1$ vectors in U is linearly dependent. True (Fundamental Th. in §4.3)
- Any set of $p+1$ vectors in U is a spanning set of U. False
- Any set of p linearly independent vectors in U is a basis of U. True (§4.3, Th 4(3))

A. true, true, true

B. false, true, true

C. true, false, true

D. true, true, false

E. true, false, false

F. false, true, false

My answer: C

6. (1 point) Which of the following are subspaces of the indicated vector spaces?

$$U = \{f \in \mathbf{F}[0,2] : f(1) = 1\}, \quad V = \{p \in \mathbb{P}_2 : p(1) = 0\}, \quad W = \{A \in \mathbf{M}_{2,2} : A^2 = A\}.$$

A. U and W only.

B. V and W only.

C. U and V only.

D. Only U.

E. Only V.

F. Only W.

U is not a subspace since it does not contain the zero function.

V is a subspace since for $p = a_0 + a_1x + a_2x^2$ we have $p(1) = a_0 + a_1 + a_2$, hence

$V = \{a_0 + a_1x + a_2x^2 : a_0 + a_1 + a_2 = 0\} = \operatorname{span}\{x-1, x^2 - x\}$

W is not a subspace since, for example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$, but $2I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin W$ because $(2I_2)^2 = 4I_2$

My answer: E

7 (6 points) For the matrix

$$A = \begin{bmatrix} 2 & -2 & 1 & 3 \\ 1 & -1 & 1 & 2 \\ 5 & -5 & 1 & 6 \end{bmatrix}$$

determine

(a) (2 points) the reduced row-echelon form of A

(b) (1 point) a basis of the row space of A,

(c) (1 point) a basis of the column space of A,

(d) (2 points) a basis of the null space of A.

$$A = \begin{bmatrix} 2 & -2 & 1 & 3 \\ 1 & - & 1 & 2 \\ 5 & -5 & & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & 3 \\ 5 & -5 & & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} & -1 & 0 & 1 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A)$$

(b) A basis of row(A) is $\{ [1 \ -1 \ 0 \ 1], [0 \ 0 \ 1 \ 1] \}$ (Th 1 in §4.4)

(c) A basis of col(A) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (Th 2 in §4.4)

The homogeneous linear system corresponding to rref(A)

$$x_1 - x_2 + x_4 = 0, \quad x_3 + x_4 = 0,$$

general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where $s, t \in \mathbb{R}$ arbitrary. A basis of the null space is therefore, (Th 6 in §4.4)

$\{ [1 \ 1 \ 0 \ 0], [-1, 0, -1, 1] \}$

8. (6 points) Find a basis and the dimension of the subspace

$$U = \{[a \quad b \quad c \quad d] \in \mathbb{R}^4 : 2a - 4b = 2c \text{ and } 3a + 5b - d = 0\}$$

of $\mathbb{R}^4$. **Justify your answer**, i.e., you must either show that you have a basis or quote some theorems from class. You do not have to show that U is a subspace.

We have $c = a - 2b$ $d = 3a + 5b$, so

$U = \{ [a, b, a-2b, 3a+5b] ; a, b \in \mathbb{R} \}$

$= \{ a[1,0,1,3] + b[0,1,-2,5] ; a, b \in \mathbb{R} \} =$

$= \text{span} \{ [1\ 0, \ ,3], [0,1,-2,5] \}$

We check that $\{ [1\ 0 \quad 3], [0,1,-2,5]$ is li

Suppose $s[1,0, \ ,3] + t[0, \quad -2,5] = [0,0,0,0]$ Since

$s[1,0,1,3] + t[0,1,-2,5] = [s, t, s-2t, 3s+5t]$

we get $s=0$ and $t=0$ by comparing the 1^{st} and 2^{nd} component.

Hence $\{ [1\ 0\ 1\ 3], [0,1,-2,5] \}$ is li and thus a basis of U

We have $\dim U =$

9. (2 points) Let A be an invertible $n \times n$ matrix. Use the Product Theorem for determinants to show that $\det(A) \neq 0$ and

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Since $AA^{-1} = I_n$ the Product Theorem for determinants gives

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$$

Therefore $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$