1. (1 point) The real part of

$$\frac{(2-3i)(4+i)}{(6-2i)(1+i)}$$

is

A. 3/4
B. 33/20
C. 3/5
D. 33/21
E. 20/33
F. 4/5

$$\frac{(2-3i)(4+i)}{(6-2i)(1+i)} = \frac{8-3i^2-i(12-2)}{6-2i^2+i(6-2)} = \frac{11-10i}{8+4i} = \frac{(11-10i)(8-4i)}{(8+4i)(8-4i)}$$

$$= \frac{88+40i^2-i(80+44)}{64+16} = \frac{1}{80}(48-124i) = \frac{3}{5} - \frac{31}{20}i$$

$\Rightarrow$ real part is $\frac{3}{5}$

My answer: C

2. (1 point) An equation of the plane passing through $A(3,-1,4)$, $B(-1,5,1)$ and $C(0,2,-2)$ is:

A. $4x-9y+36z=18$
B. $9x+5y-2z=14$
C. $7x-8y+5z=6$
D. $8x-11y+18z=24$
E. $3x-2y+z=0$
F. $3x+2y-z=0$

Let $ax+by+cz=d$ be an eq of the plane. Since the coordinates of A, B, C satisfy this eq, we get the following linear system for a, b, c

$$\left\{\begin{array}{l} 3a-b+4c=d \\ -a+5b+c=d \\ 2b-2c=d \end{array}\right\} \text{ augmented matrix is } \begin{bmatrix} 3 & -1 & 4 & d \\ -1 & 5 & 1 & d \\ 0 & 2 & -2 & d \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & -1 & -d \\ 0 & 14 & 7 & 4d \\ 0 & 2 & -2 & d \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -1 & -d \\ 0 & 1 & -1 & d/2 \\ 0 & 0 & 21 & -3d \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 0 & -(1+1/7)d \\ 0 & 1 & 0 & (\frac{1}{2}-\frac{1}{7})d \\ 0 & 0 & 1 & -d/7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 0 & -8/7\, d \\ 0 & 1 & 0 & 5/14\, d \\ 0 & 0 & 1 & -d/7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & (-8/7+25/14)d \\ 0 & 1 & 0 & 5/14\, d \\ 0 & 0 & 1 & -d/7 \end{bmatrix} \Rightarrow \begin{array}{l} a = \frac{25-16}{14}d = \frac{9}{14}d \\ b = 5/14\, d \\ c = -\frac{1}{7}d \end{array}$$

Hence, for $d=14$ we get $a=9$, $b=5$, $c=-2$.

My answer: B

3. (1 point) Among the matrices

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 & 2 & -3 & 5 \\ 0 & 0 & 3 & 0 & -2 \\ 1 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}.$$

= ref = rref

the following are in row-echelon form, but not in reduced row-echelon form

A. A, C and E.
B. A, B and D.
C. A, B and C.
D. B and C only.
E. A and E only.
F. B and E only.

A ref, not rref because of (13)-position
B not ref because of 2nd row
C rref
D not ref because of 1st/2nd rows
E ref, not rref because of (12)-position

My answer: E

4. (1 point) If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and B is a $2 \times n$ matrix then the second row of the matrix AB is

A. not defined unless $n = 2$.
B. the same as the second row of B.
C. the same as the second row of A.
D. the same as the first row of B.
E. the same as the first row of A.
F. the sum of the first and the second row of B.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{11}+b_{12} & b_{12}+b_{22} & \cdots & b_{1n}+b_{2n} \end{bmatrix}$$

My answer: F

5. (1 point) Compute the determinant $\begin{vmatrix} 0 & 0 & 0 & 7 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix}$.

A. -25
B. -21
C. 33
D. -41
E. -35
F. 45

$$\underset{1^{st}\text{ row}}{=} -7\begin{vmatrix} 3 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} \underset{(3^{rd}\text{ row})}{=} -(-7)\begin{vmatrix} 3 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \underset{3^{rd}\text{ row}}{=}$$

$$= -7\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = -7(3+2) = -35$$

My answer: E

6. (1 point) Which combination of true/false is correct for the following statements for $n \times n$ matrices A and B:

- If $A \neq 0$ then A is invertible. False, e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible
- If A is invertible then $A \neq 0$. True since 0 is not invertible
- If $AB = I_n$ then A is invertible and $B = A^{-1}$. True (Invertible Matrix Theorem)

A. true, false, true
B. false, true, false
C. true, true, false
D. false, false, false
E. true, false, false
F. false, true, true

My answer: F

7. (6 points) (no partial credit) Suppose a row-echelon form of a 4×4 matrix A has 2 leading **1**'s. Answer the following questions.

(a) (1 point) Is the system $AX = B$ consistent for any choice of vectors B in $\mathbb{R}^4$? No

The matrix A has rank 2, hence is not invertible. Because of the IMT (= Invertible Matrix Theorem), $AX = B$ is not solvable for all $B \in \mathbb{R}^n$

(b) (1 point) If $AX = B$ is consistent, will there be infinitely many solutions? Yes,

the general solution will have $n - r = 4 - 2 = 2$ free parameters (Th3 in §1.2)

(c) (1 point) Does the system $AX = 0$ have a unique solution? No,

Reason: Same as for (b), or because A is not invertible.

(d) (1 point) If B is a linear combination of the columns of A, is then the linear system $AX = B$ solvable?

Yes, since $B = c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_3$ (for A_i = columns of A) $\iff$ $A \begin{bmatrix} c_1 \\ \vdots \\ c_4 \end{bmatrix} = B$ (see §1.4).

(e) (1 point) Is A^T invertible? No, since A invertible $\iff$ A^T invertible (§1.5.4, Cor. to Th3) and A is not invertible because rank $A = 2 < 4$.

(f) (1 point) Is $\det(A) \neq 0$? No,

since $\det(A) \neq 0 \iff A$ is invertible.

8. **(8 points)** Consider the system of linear equations

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 0 \\ kx & + & 4y & + & z & = & 12 \\ -x & - & 2y & + & kz & = & 4 \end{array}$$

where k is a real parameter. Find the conditions on k, so that the system has
(i) infinitely many solutions,
(ii) a unique solution, and
(iii) no solution.
In case (i) write done all solutions.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ k & 4 & 1 & 12 \\ -1 & -2 & k & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 4-2k & 1+k & 12 \\ 0 & 0 & k-1 & 4 \end{bmatrix} =: A'$$

Case (1) $4-2k=0$, i.e. $k=2$

$$A' = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{array}{l} x = 4-2t \\ y = t \text{ free parameter} \\ z = 4 \end{array}$$

Case (2) $k=1$:
$$A' = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 2 & 12 \\ 0 & 0 & 0 & 4 \end{bmatrix} \Rightarrow \text{no solution}$$

A' :

Case (3) $k \neq 2$ and $k \neq 1$:

$$A' \sim \begin{bmatrix} 1 & 2 & -1 & \\ 0 & 1 & \frac{k+1}{4-2k} & \frac{12}{4-2k} \\ 0 & 0 & 1 & \frac{4}{k-1} \end{bmatrix} \Rightarrow \text{uniquely solvable.}$$

Hence, we have

(i) $\Leftrightarrow k=2$, (ii) $\Leftrightarrow k \neq 2$ and $k \neq 1$, (iii) $\Leftrightarrow k=1$

and in case (i) all solutions are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4-2t \\ t \\ 4 \end{bmatrix}, \quad t \text{ a free parameter}$$