1. (a) (2 points) Show that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is a basis of $\mathbb{M}_{22}$.

$\{A_1, A_2, A_3, A_4\}$ is li since $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = 0 \iff$

$\begin{bmatrix} c_1+c_2 & c_3+c_4 \\ c_3-c_4 & c_1-c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff \begin{matrix} c_1+c_2 = 0 = c_1-c_2 \\ c_3+c_4 = 0 = c_3-c_4 \end{matrix} \iff \begin{matrix} c_1 = 0 = c_2 \\ c_3 = 0 = c_4 . \end{matrix}$

Since $\dim \mathbb{M}_{22} = 4$, $\{A_1, A_2, A_3, A_4\}$ is a basis by Th.4 of §5.2

(Alternative solution: Show $\{A_1, \dots, A_4\}$ is a spanning set and apply Th 4 of §5.2)

(b) (2 points) If $\{v_1, v_2, v_3, \dots, v_n\}$, $n > 1$, is a linearly independent subset of a vector space V, then $\{v_2, \dots, v_n\}$ is linearly independent too.

Suppose $c_2v_2 + \dots + c_nv_n = 0$ for scalars $c_2, \dots, c_n \in \mathbb{R}$. Then also $0v_1 + c_2v_2 + \dots + c_nv_n = 0$. Since $\{v_1, \dots, v_n\}$ is li, all coefficients are zero, so $c_2 = 0 = c_3 = \dots = c_n$.

(c) (2 points) Let $A \in \mathbb{M}_{nn}$ be a matrix with $A^3 = 0$, but $A^2 \neq 0$. Show that then $\{I, A, A^2\}$ is linearly independent, where I is the $n \times n$-identity matrix.

Suppose $c_0I + c_1A + c_2A^2 = 0$. Multiply by A^2 to get $0 = A^2(c_0I + c_1A + c_2A^2) = c_0A^2 + c_1A^3 + c_2A^4 = c_0A^2$. Since $A^2 \neq 0$ this forces $c_0 = 0$. Thus $c_1A + c_2A^2 = 0$. Now multiply by A and get $0 = A(c_1A + c_2A^2) = c_1A^2 + c_2A^3 = c_1A^2$. Since $A^2 \neq 0$ this forces $c_1 = 0$. Thus $0 = c_2A^2$. Again $A^2 \neq 0$ implies $c_2 = 0$. Since $c_0 = c_1 = c_2 = 0$, $\{I, A, A^2\}$ is li

2. (a) (2 points) Show that $U = \{A \in \mathbb{M}_{nn} : A^t = -A\}$ is a subspace of $\mathbb{M}_{nn}$.
(b) (4 points) Find a basis of the subspace U and determine its dimension.

(a) We do the Subspace Test (Th3 in §5.1):

(1) $0 \in M_{nn}$ since $0^T = 0 = -0$

(2) Let $A, B \in U$, i.e. $A^T = -A$ and $B^T = -B$. Then $(A+B)^T = A^T + B^T = (-A) + (-B) = -(A+B)$ shows $A+B \in U$

(3) Let $A \in U$, i.e., $A^T = -A$, and let $s \in \mathbb{R}$. Then $(sA)^T = sA^T = -sA$ shows $sA \in U$.

(b) A matrix $A = [a_{ij}]$ lies in U if and only if $a_{ij} = -a_{ji}$ for all $1 \le i, j \le n$. For $1 \le i < j \le n$ define

$$F_{ij} = \begin{array}{c} \\ \left[\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}\right] \end{array} \begin{array}{l} \leftarrow i \\ \leftarrow j \end{array},$$

i.e. the (pq)-coefficient of F_{ij} is $\begin{cases} 1 & \text{for } (pq) = (ij) \\ -1 & \text{for } (pq) = (ji) \\ 0 & \text{otherwise} \end{cases}$

Then $A \in U \Leftrightarrow A = \sum_{1 \le i < j \le n} a_{ij} F_{ij}$. Thus $U = \operatorname{span}\{F_{ij} : 1 \le i < j \le n\}$. The set $\{F_{ij} : 1 \le i < j \le n\}$ is li: If $\sum_{1 \le i < j \le n} c_{ij} F_{ij} = 0$, the (pq)-coefficient, $1 \le p < q \le n$, on the left side is c_{pq}, while the (pq)-coeff. on the right side is 0. So all $c_{ij} = 0$. This proves that $\{F_{ij} : 1 \le i < j \le n\}$ is a basis of U.

(1p) There are $(n-1) + (n-2) + \dots + 2 + 1 = \sum_{p=1}^{n-1} p = \frac{1}{2} n(n-1)$ elements in this basis of U. Hence $\dim U = \frac{1}{2} n(n-1)$.

Solution of 3(c): Let a be the sequence with $a_k = 3^k + (-1)^k 3$ and b the sequence with $b_k = 3^k - (-1)^k$. Then $V = \operatorname{span}\{a, b\}$. The sequences a, b are li: If $sa + tb = 0$ then $3^k(s+t) + (-1)^k(3s - t) = 0$ follows for all k. For $k = 0$ we get $3^0 = 1$, so $s + t + 3s - t = 4s = 0$, hence $s = 0$. For $k = 1$ we then get $3t - (-t) = 4t = 0$, so $t = 0$. Thus $\{a, b\}$ is a basis of V, and $\dim V = 2$ follows.

3. Let $\mathbb{S}$ be the set of all infinite sequences $a = (a_0, a_1, \dots)$ of real numbers. For $a = (a_0, a_1, \cdots)$ and $b = (b_0, b_1, \cdots)$ and $r \in \mathbb{R}$ define equality, addition and scalar multiplication of sequences as follows:

$$\begin{aligned} a = b &\iff a_i = b_i \text{ for all } i, \text{ i.e., } a_0 = b_0, a_1 = b_1 \text{ etc.,} \\ a + b &= (a_0 + b_0, a_1 + b_1, a_2 + b_2, \cdots), \\ ra &= (ra_0, ra_1, \cdots) \end{aligned}$$

One can check that with these operations $\mathbb{S}$ is a vector space.

(a) (? points) Show that $U = \{a = (a_0, a_1, \cdots) : a_i = 0$ for all but finitely many $i\}$ is an infinite dimensional subspace of $\mathbb{S}$. Note that the condition defining U says that $a_n = 0$ from a certain N on, i.e., for $n \geq N$. However, the N depends on the sequence a. It is not the same for all sequences!

(b) (2 points) Show that the set V of all solutions of the recurrence relation $x_{k+2} = 3x_k + 2x_{k+1}$, is a 2-dimensional subspace of $\mathbb{S}$. You may use without proof (see problem 6 of assignment 2) that the general solution of the recurrence relation $x_{k+2} = 3x_k + 2x_{k+1}$ is given by $x = (x_0, x_1, \cdots)$ where

$$x_k = 3^k(s+t) + (-1)^k(3s - t), \quad s, t \in \mathbb{R} \text{ arbitrary.}$$

(a) We apply the subspace test: (1) The zero in $\mathbb{S}$ is the sequence $0 = (0, \dots, 0)$, i.e. all terms of the sequence are zero. This lies in U.

(2) Let $a, b \in U$. Then there exists N s.th. $a_n = 0 = b_n$ for all $n \geq N$. Hence also $a_n + b_n = 0$ for $n \geq N$, proving $a + b \in U$

(3) Let $a \in U$ and let $r \in \mathbb{R}$. Since $a_n = 0$ for $n \geq N$ (for a suitable N) we have $ra_n = 0$ for $n \geq N$. Thus $ra \in U$.

We have now checked that U is a subspace.

(b) To show that U is infinite dimensional, it suffices to show that for any $n \in \mathbb{N}$ there are n li sequences in U.

Define e_i is the sequence whose n^{th} term is 1 if $n = i$ and is 0 otherwise. Then $\{e_1, \dots, e_n\}$ is li since whenever $c_1 e_1 + \dots + c_n e_n = 0$ the p^{th} component $(1 \leq p \leq n)$ of the left side is c_p, so $c_p = 0$ follows.

Other solution: Suppose U is spanned by sequences $a^{(1)}, a^{(2)}, \dots, a^{(n)}$. For each i there is N_i s.th. $a^{(i)}_n = 0$ for $n \geq N_i$. Let $N = \max\{N_1, \dots, N_n\}$. Then any $b \in \operatorname{span}\{a^{(1)}, \dots, a^{(n)}\}$ has the property that $b_p = 0$ for $p \geq N$. Hence the sequence e_{N+1} (see above) does not lie in U. This shows that U does not have a finite spanning set.

4. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (a) (1 point) Show that $T : \mathbb{M}_{22} \to \mathbb{M}_{22}$ given by $T(X) = AX - XA$ is a linear map.
(b) (2 points) Find a basis of the kernel of T.
(c) (1 point) Find the dimension of the image of T.

(a) For $X_1, X_2 \in \mathbb{M}_{22}$ we have $T(X_1+X_2) = A(X_1+X_2) - (X_1+X_2)A = AX_1 + AX_2 - X_1A - X_2A = (AX_1 - X_1A) + (AX_2 - X_2A) = T(X_1) + T(X_2)$. For $r \in \mathbb{R}$ and $X \in \mathbb{M}_{22}$ we have $T(rX) = A(rX) - (rX)A = r(AX - XA) = rT(X)$.
Thus T is linear. (Note this works for any matrix A!)

(b) For $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we calculate

$$T(X) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} - \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix}$$

Hence

$$X \in \operatorname{Ker} T \iff T(X) = 0 \iff b = c,\ a = d \quad \text{for } X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\iff X = \begin{bmatrix} s & t \\ t & s \end{bmatrix} \text{ for } s, t \in \mathbb{R}$$
$$\iff X = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus $\operatorname{Ker} T = \operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$. Since $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is part of a basis of $\mathbb{M}_{22}$ (problem 1a), it is li, so $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis of $\operatorname{Ker} T$.

(c) By the Dimension Theorem we have $\dim \operatorname{Im} T = \dim \mathbb{M}_{22} - \dim \operatorname{Ker} T = 4 - 2 = 2$.

Alternative solution: From the calculation in (b) we get

$$Y = \begin{bmatrix} u & v \\ w & z \end{bmatrix} \in \operatorname{Im} T \iff Y = T(X) \text{ for some } X \iff \begin{bmatrix} u & v \\ w & z \end{bmatrix} = \begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix}$$

for suitable $a, b, c, d \iff Y = \begin{bmatrix} u & v \\ -v & -u \end{bmatrix} = u\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + v\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

It then follows as in (b) that $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis of $\operatorname{Im} T$, so $\dim \operatorname{Im} T = 2$.