University of Ottawa

Department of Mathematics and Statistics

MAT 1341: Introduction to Linear Algebra

Instructor: Erhard Neher

Assignment 3; due July 10, 2006, 17:00 in the class room

Family Name: ______________________

First Name: ______________________

Student number: ______________________

Please read these instructions carefully:

- Enter your name on this page and the next, but your student number only on this page. You will get back the assignment without this first page.
- The table below is for the TA. Do not write in the table. For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages of the assignment. Therefore, **fill in your name on both pages** and your student number on this page only.

Quest.	1	2	3	4	Total
maximal	6	6	7	5	24
score					

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Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
- For all questions you must show your work to obtain the points. Simply writing the correct answer will earn you 0.
- Please write legibly and argue logically: You must convince the TA that you know why your solution is correct.
- You have to submit this assignment at the beginning of the DGD on Monday, July 10, at 17:00 in the classroom at the latest. If you wish to submit it earlier, please do so at the secretariat of the Department of Mathematics, room 103A, 8:45–12:00 and 13:00–17:00.

1. (a) (2 points) Show that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is a basis of $\mathbb{M}_{22}$.

(b) (2 points) If $\{v_1, v_2, v_3, \ldots, v_n\}$, $n > 1$, is a linearly independent subset of a vector space V, then $\{v_2, \ldots, v_n\}$ is linearly independent too.

(c) (2 points) Let $A \in \mathbb{M}_{nn}$ be a matrix with $A^3 = 0$, but $A^2 \neq 0$. Show that then $\{I, A, A^2\}$ is linearly independent, where I is the $n \times n$-identity matrix.

2. (a) (2 points) Show that $U = \{A \in \mathbb{M}_{nn} : A^t = -A\}$ is a subspace of $\mathbb{M}_{nn}$.
(b) (4 points) Find a basis of the subspace U and determine its dimension.

3. Let $\mathbb{S}$ be the set of all infinite sequences $a = (a_0, a_1, \cdots)$ of real numbers. For $a = (a_0, a_1, \cdots)$, $b = (b_0, b_1, \cdots) \in \mathbb{S}$ and $r \in \mathbb{R}$ define equality, addition and scalar multiplication of sequences as follows:

$$\begin{aligned} a = b &\iff a_i = b_i \text{ for all } i, \text{ i.e., } a_0 = b_0, a_1 = b_1 \text{ etc.,} \\ a + b &= (a_0 + b_0, a_1 + b_1, a_2 + b_2, \cdots), \\ ra &= (ra_0, ra_1, ra_2, \cdots) \end{aligned}$$

One can check that with these operations $\mathbb{S}$ is a vector space.

(a) (2 points) Show that $U = \{a = (a_0, a_1, \cdots) : a_i = 0 \text{ for all but finitely many } i\}$ is a subspace of $\mathbb{S}$. Note that the condition defining U says that $a_n = 0$ from a certain N on, i.e., for $n \geq N$. However, the N depends on the sequence a. It is not the same for all sequences!

(b) (3 points) Show that U is infinite dimensional.

(c) (2 points) Show that the set V of all solutions of the recurrence relation $x_{k+2} = 3x_k + 2x_{k+1}$, is a 2-dimensional subspace of $\mathbb{S}$. You may use without proof (see problem 6 of assignment 2) that the general solution of the recurrence relation $x_{k+2} = 3x_k + 2x_{k+1}$ is given by $x = (x_0, x_1, \cdots)$ where

$$x_k = 3^k(s + t) + (-1)^k(3s - t), \quad s, t \in \mathbb{R} \text{ arbitrary.}$$

4. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (a) (1 point) Show that $T : \mathbb{M}_{22} \to \mathbb{M}_{22}$ given by $T(X) = AX - XA$ is a linear map.
(b) (3 points) Find a basis of the kernel of T.
(c) (1 point) Find the dimension of the image of T.