1. (3 points) Let $x_1, x_2, \ldots, x_n$ be real numbers, $n \geq 2$. Show that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\leq i<j\leq n} (x_j - x_i)$$

where $\prod_{1\leq i<j\leq n}(x_j - x_i)$ means the product of all factors $(x_j - x_i)$ for all pairs (i,j) satisfying $i < j$ and i and j between 1 and n. Hint: Replace the i^{th}-column C_i of the matrix in question by $C_i - x_1 C_{i-1}$ and expand along the first row.

$$\det = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & x_2-x_1 & x_2^2-x_2x_1 & x_2^3-x_2^2x_1 & \cdots & x_2^{n-1}-x_2^{n-2}x_1 \\ 1 & x_3-x_1 & x_3^2-x_3x_1 & x_3^3-x_3^2x_1 & & x_3^{n-1}-x_3^{n-2}x_1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n-x_1 & x_n^2-x_nx_1 & x_n^3-x_n^2x_1 & & x_n^{n-1}-x_n^{n-2}x_1 \end{vmatrix} =$$

$$= \begin{vmatrix} x_2-x_1 & x_2(x_2-x_1) & x_2^2(x_2-x_1) & \cdots & x_2^{n-2}(x_2-x_1) \\ x_3-x_1 & x_3(x_3-x_1) & x_3^2(x_3-x_1) & \cdots & x_3^{n-2}(x_3-x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ x_n-x_1 & x_n(x_n-x_1) & x_n^2(x_n-x_1) & \cdots & x_n^{n-2}(x_n-x_1) \end{vmatrix} =$$

(pull out factors $x_i - x_1$ in column C_i)

$$= (x_2-x_1)(x_3-x_1)\cdots(x_n-x_1) \underbrace{\begin{vmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-2} \\ \vdots & \vdots & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{vmatrix}}_{= \prod_{2\leq i<j\leq n}(x_j - x_i)}$$

$$= \prod_{1\leq i<j\leq n}(x_j - x_i)$$

2. (3 points) Solve the following linear system using Gaussian elimination

$$\begin{array}{rcrcl} (1+i)x & + & (2+i)y & = & 5 \\ (2-2i)x & + & iy & = & 1+2i \end{array}$$

Solution

augmented matrix $\begin{bmatrix} 1+i & 2+i & 5 \\ 2-2i & i & 1+2i \end{bmatrix} \underset{R_1 \rightsquigarrow \frac{1}{(1+i)} R_1}{\sim} \begin{bmatrix} 1 & \frac{3}{2} - \frac{i}{2} & \frac{5}{2} - \frac{5}{2} i \\ 2-2i & i & 1+2i \end{bmatrix}$

$$\underset{R_2 \rightsquigarrow R_2 - (2-2i) R_1}{\sim} \begin{bmatrix} 1 & \frac{3}{2} - \frac{i}{2} & 5/2 - 5/2\, i \\ 0 & -2+5i & 1+12i \end{bmatrix}$$

$$\underset{R_2 \rightsquigarrow \frac{1}{-2+5i} R_2}{\sim} \begin{bmatrix} 1 & 3/2 - i/2 & 5/2 - 5/2\, i \\ 0 & 1 & 2-i \end{bmatrix}$$

$$\underset{R_1 \rightsquigarrow R_1 - (\frac{3}{2} - \frac{1}{2} i) R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2-i \end{bmatrix}$$

Hence the solution is $x = 0$, $y = 2-i$

3. (3 points) Find the solution of the following system of linear differential equations

$$\begin{array}{rcrcl} f_1 & + & 3f_2 & = & f_1', \\ 2f_1 & + & 2f_2 & = & f_2', \end{array} \qquad \begin{array}{l} f_1(0) = -1, \\ f_2(0) = 8. \end{array}$$

Let $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. Thus $Af = f'$ where $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. The characteristic polynomial of A is $\begin{vmatrix} \lambda-1 & -3 \\ -2 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$

Thus A is diagonalizable with eigenvalues 4 and -1. We calculate basic eigenvectors:

$\lambda = 4$: $\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $E_4(A) = \mathbb{R}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$ $\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$, $E_1(A) = \mathbb{R}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

The general solution is therefore (Th 1 in §2.8)

$$f(x) = c_1 e^{4x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-x}\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad c_i \in \mathbb{R}$$

We determine c_1, c_2 using the initial condition $f(0) = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$

$f(0) = \begin{bmatrix} c_1 + 3c_2 \\ c_1 - 2c_2 \end{bmatrix}$, so $\begin{array}{l} c_1 + 3c_2 = -1 \\ c_1 - 2c_2 = 8 \end{array}$. The unique solution

to this system is $c_1 = \frac{22}{5}$, $c_2 = -9/5$. Hence the solution is

$$f(x) = \frac{22}{5} e^{4x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{9}{5} e^{-x}\begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

4. (3 points) Are the following subspaces of $\mathbb{R}^3$? Provide a short justification!

(a) $U = \{(x, y, z) \in \mathbb{R}^3 : 2x - 3y = 0, y + 4z = 0\}$,

Yes, since $U = \text{null } A$ for $A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

(b) $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 = y\overset{2}{\cancel{z}}\}$,

V, since V is not closed under addition: $(1,-1,0) \in V$ and $(1,1,0) \in V$ but $(1,-1,0)+(1,1,0) = (2,0,0) \notin V$

(c) $W = \left\{ \begin{bmatrix} 2s - t \\ 3s + t \\ 4t \end{bmatrix} : s, t \in \mathbb{R} \right\}$.

Yes, since $W = \left\{ s\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, s,t \in \mathbb{R} \right\}$

$= \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\}$.

5. (3 points) In each case either show that the statement is true or give an example showing that it is false. Throughout X_1, X_2 are vectors in $\mathbb{R}^n$.

(a) If $\{X_1, X_2\}$ is linearly independent, then so is $\{X_1, X_2, X_1 - X_2\}$. False,

since $-X_1 + X_2 + (X_1 - X_2) = 0$ is linear dependence relation

(b) If $\{X_1, X_2\}$ is linearly dependent, then so is $\{X_1, X_2, X_1 - X_2\}$. True

Reason: There exist $c_1, c_2 \in \mathbb{R}$, not both 0, such that $c_1X_1 + c_2X_2 = 0$

Then $c_1X_1 + c_2X_2 + 0(X_1 - X_2) = 0$ is a linear dependence relation

for $\{X_1, X_2, X_1 - X_2\}$.

(c) If $\{X_1, X_2\}$ is linearly independent, then $c_1X_1 + c_2X_2 = 0$ for some $c_1, c_2 \in \mathbb{R}$. True

We always have $0X_1 + 0X_2 = 0$, so we can take $c_1 = 0 = c_2$.

(This holds for any set $\{X_1, X_2\}$!)

6. (a) Diagonalize the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

(b) Solve the recurrence relation

$$x_{k+2} = 3x_k + 2x_{k+1}, \qquad x_0 = 1 = x_1,$$

i.e., find a closed form expression for x_k.

(a) Characteristic polynomial $c_A(\lambda) = \begin{vmatrix} \lambda & -1 \\ -3 & \lambda-2 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$, so the e-values are $\lambda = 3$ and $\lambda = -1$

$E_3(A)$: $3I_2 - A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a basic e-vector

$E_{-1}(A)$: $-I_2 - A = \begin{bmatrix} -1 & -1 \\ -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a basic e-vector

Let $P = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$; $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$, then $A = PDP^{-1}$.

(b) Let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. Then $V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 3x_k + 2x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = AV_k$.

Let $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = P^{-1}V_0 = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then

$$V_k = AV_{k-1} = \cdots = A^k V_0 = (PDP^{-1})^k V_0 = PD^kP^{-1}V_0 =$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3^k \\ (-1)^k \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 3^k + (-1)^k \\ 3^{k+1} + (-1)^{k+1} \end{bmatrix}$$

Since $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ this proves $x_k = \frac{1}{2}\left(3^k + (-1)^k\right)$

7. Let A be an $n \times n$ matrix. (a) **(1 point)** If λ is an eigenvalue of A show that $3\lambda^2 - 3\lambda + 5$ is an eigenvalue of $3A^2 - 3A + 5I_n$ where I_n is the $n \times n$ Identity matrix.

(b) **(2 points)** Suppose $A^2 = 0 \neq A$. Show that $\lambda = 0$ is the only eigenvalue of A. (You have to show that (i) if λ is an eigenvalue of A then necessarily $\lambda = 0$, and (ii) $\lambda = 0$ is an eigenvalue of A)

(a) Since λ is an e-value there exists $0 \neq X \in \mathbb{R}^n$ s.th. $AX = \lambda X$. For this X we get $(3A^2 - 3A + 5I_n)X = 3A^2X - 3AX + 5I_nX = 3A(AX) - 3\lambda X + 5X =$ $= 3\lambda(AX) - 3\lambda X + 5X = 3\lambda^2 X - 3\lambda X + 5X = (3\lambda^2 - 3\lambda + 5)X$, so X is an e-vector of $3A^2 - 3A + 5I_n$ with e-value $3\lambda^2 - 3\lambda + 5$.

(b.i) Let λ be an e-value of A. Then $AX = \lambda X$ for some $X \neq 0$. Hence $0 = A^2X = A(AX) = \lambda AX = \lambda^2 X$. Since $X \neq 0$ this implies $\lambda^2 = 0$, so $\lambda = 0$.

(b.ii) Since $A \neq 0$ there exists $Y \in \mathbb{R}^n$ s.th. $X = AY \neq 0$. Then $AX = A^2Y = 0$ shows that X is an e-vector of A with e-value 0.

(other solution) $\det(A^2) = \det(0) = 0$; since $\det(A^2) = \det(A)^2$ this proves that $\det(A) = 0$, ie $\lambda = 0$ is a solution of the characteristic eq $c_A(\lambda) = \det(\lambda I_n - A)$ — — use $\det(-A) =$ $= (-1)^n \det A$!!)