1. (1 point) Calculate all possible products between the matrices

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1+4-3 & 0+6-12 \\ 0+6+2 & 0+9+8 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 8 & 17 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 0 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -1+0 & -2+0 & 3+0 \\ 2+0 & 4+9 & -6+6 \\ 1+0 & 2+12 & -3+8 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 3 \\ 2 & 13 & 0 \\ 1 & 14 & 5 \end{bmatrix}$$

My answer: $\begin{bmatrix} 0 & -6 \\ 8 & 17 \end{bmatrix}, \begin{bmatrix} -1 & -2 & 3 \\ 2 & 13 & 0 \\ 1 & 14 & 5 \end{bmatrix}$

2. (1 point) Let A and B be arbitrary matrices. Which combination of true/false is correct for the following statements:

- If $AB = BA$ then A and B are both square and of the same size. True
- If A has a row of zeros then also AB has a row of zeros. True
- $(A+B)^2 = A^2 + 2AB + B^2$ always holds. False

(A) true, false, true
~~(B) false, true, false~~
(C) true, true, false
~~(D) true, false, false~~
(E) false, false, true
(F) false, false, false

My answer: (C)

- A $m \times n$, B $p \times q$; AB exists $\Rightarrow n = p$, BA exists $\Rightarrow q = m$, so AB is $m \times m$ and BA is $n \times n$. Since $AB = BA \Rightarrow m = n$ True
- If the ith row of A is zero then any entry in the ith row of AB is (ith row of A) $\cdot$ (jth column of B) $= 0 \cdot$ (column of B) $= 0$ True
- $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$ is true. Hence $(A+B)^2 = A^2 + 2AB + B^2 \iff AB + BA = 2AB \iff AB = BA$, which is not always true; False!

3. (1 point) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$

Apply inversion algorithm:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 2 & 3 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 2 & -6 \\ 0 & 1 & 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right]$$

My answer: $A^{-1} = \begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & 2 \end{bmatrix}$

4. (1 point) Find A if

$$\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}.$$

equation $\iff$ $A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 8 \end{bmatrix}$

$\iff A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 8 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 0 \\ 3 & 6 & 12 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 3 \\ -2 & -1 & -4 \end{bmatrix}$

My answer: $A = \begin{bmatrix} -1 & 3 & 3 \\ -2 & -1 & -4 \end{bmatrix}$

5. (1 point) Find basic solutions for the homogeneous linear system whose coefficient matrix is given below and express the general solution as a linear combination of these basic solutions.

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 2 & -1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 2 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

corresponding linear system

$$x_1 + 2x_2 - 7x_5 = 0$$
$$x_3 - 3x_5 = 0$$
$$x_4 + x_5 = 0$$

general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 7t \\ s \\ 3t \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

basic solutions

My answer: ____________________

6. (1 point) Suppose A is a 3×3-matrix and $AX = 0$ has a unique solution. Which combination of true/false is correct for the following statements:

- The reduced row echelon form of A is the 3×3 identity matrix I_3. True
- There may exist column vectors B for which the equation $AX = B$ is not consistent. False
- The equation $AX = B$ is uniquely solvable for every column vector $B \in \mathbb{R}^3$. True

(A) true, false, true
(B) false, true, false
(C) true, true, false
(D) true, false, false
(E) false, false, true
(F) false, false, false

My answer: (A)

Unique solution of $AX = 0 \Rightarrow \text{rref } A = I_3$ (Invertible Matrix Th)

$\Rightarrow$ $AX = B$ is unique solvable for every B (also Invertible Matrix Th)

7. (8 points) Consider the system of linear equations

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & -1 \\ x & + & 2y & + & kz & = & 0 \\ -x & + & ky & + & 2z & = & 0 \end{array}$$

where k is a real parameter. Find the conditions on k, so that the system has
(i) infinitely many solutions,
(ii) a unique solution, and
(iii) no solution.
In the cases (i) and (ii) write done all respectively the unique solution.

augmented matrix is $A = \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 2 & k & 0 \\ -1 & k & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & k-2 & 1 \\ 0 & k+1 & 4 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & k-2 & 1 \\ 0 & 0 & * & -(k+2) \end{array}\right]$

where $* = 4-(k+1)(k-2) = 4-(k^2-k-2) = -(k^2-k-6) = -(k-3)(k+2)$

Case (i) $k+2=0$, i.e. $k=-2$. Then

$A \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 6 & -2 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, i.e. $\begin{array}{l} x+6z=-2 \\ y-4z=1 \end{array}$

general sol is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2-6t \\ 1+4t \\ t \end{bmatrix}$, $t \in \mathbb{R}$ arbitrary, ∞-many sol.

Case (ii) $(k-3)(k+2) \neq 0$, i.e. $k \neq -2$ and $k \neq 3$. Then

$A \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & k-2 & 1 \\ 0 & 0 & (k-3)(k+2) & k+2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & k-2 & 1 \\ 0 & 0 & 1 & \frac{1}{k-3} \end{array}\right]$

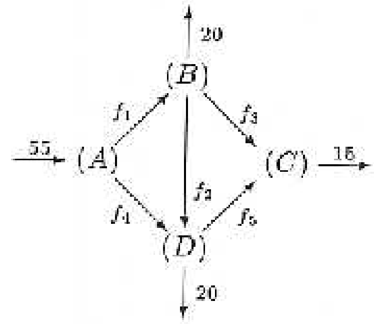
Hence $z = \frac{1}{k-3}$, $y = 1 - \frac{k-2}{k-3} = \frac{k-3-k+2}{k-3} = -\frac{1}{k-3}$

$x = -1-y-2z = \frac{1}{k-3}\left(-k+3+1-2\right) = -\frac{k-2}{k-3}$, unique sol.

Case (iii) $k=3$:

$A \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow$ not solvable

8. (6 points) Consider the following network of irrigation channels.



(a) **(1 point)** Write down a system of linear equations in the variables $f_1, \ldots, f_5$ that describes the flow in the network.

(b) **(3 points)** Solve the system in (a). Show all details of your solution!

(c) **(2 points)** If the canal BC is closed, what range of flow on AD must be maintained so that no canal carries a flow of more than 30?

(a)
$$f_1 + f_4 = 55$$
$$f_1 - f_2 - f_3 = 20$$
$$f_3 + f_5 = 15$$
$$f_2 + f_4 - f_5 = 20$$

(b)
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 55 \\ 1 & -1 & -1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & -1 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 55 \\ 0 & +1 & +1 & +1 & 0 & +35 \\ 0 & 0 & 1 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & -1 & 20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 55 \\ 0 & 1 & 1 & 1 & 0 & 35 \\ 0 & 0 & 1 & 0 & 1 & 15 \\ 0 & 0 & -1 & 0 & -1 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 55 \\ 0 & 1 & 1 & 1 & 0 & 35 \\ 0 & 0 & 1 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 55 \\ 0 & 1 & 0 & 1 & -1 & 20 \\ 0 & 0 & 1 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} f_1 = 55 - f_4 \\ f_2 = 20 - f_4 + f_5 \\ f_3 = 15 - f_5 \end{array}$$

f_4, f_5 free parameters

(c) BC closed $\Leftrightarrow f_3 = 0 \Leftrightarrow f_5 = 15$.

$$f_2 = 35 - f_4 \le 30 \Leftrightarrow f_4 \ge 5$$
$$f_1 = 55 - f_4 \le 30 \Leftrightarrow 25 \le f_4$$

Hence on AB $= f_4$ one must maintain $25 \le f_4 \le 30$.