1. (1 point) Let $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$. If $(2X + A^T)^T = 2A$ then the sum of the entries on the diagonal of X is

A. -1
B. 4
C. 2
D. 0
E. 10
F. 3

$(2X+A^T)^T = 2A \Leftrightarrow 2X + A^T = 2A^T \Leftrightarrow 2X = 2A^T - A^T = A^T \Leftrightarrow$

$X = \frac{1}{2} A^T = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}$, so $\frac{1}{2}(2+2) = 2$

My answer: C

2. (1 point) Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$. The first row of A^{-1} is

A. $[1 \quad -4 \quad -9]$
B. $[3 \quad 0 \quad 4]$
C. $[2 \quad -3 \quad 14]$
D. $[1 \quad 4 \quad 7]$
E. $[2 \quad 3 \quad 0]$
F. $[1 \quad -5 \quad 6]$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 2 & 5 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 7 & -14 \\ 0 & 1 & 0 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -5 & 6 \\ 0 & 1 & 0 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 1 & -5 & 6 \\ 0 & 3 & -5 \\ 0 & -1 & 2 \end{bmatrix}$

My answer: F

3. (1 point) Calculate the determinant

$$\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 3 \\ c & 0 & 0 & d \\ e & f & 4 & g \end{vmatrix}$$

A. $4acfg$

B. $-abcf$

C. $3abcfg$

D. $-3abcfg$

E. $12abc$

F. $-abcfg$

$\underset{\text{1st row}}{=} -a \begin{vmatrix} 0 & 0 & b \\ c & 0 & 0 \\ e & f & 4 \end{vmatrix} \underset{\text{1st row}}{=} -ab \begin{vmatrix} c & 0 \\ e & f \end{vmatrix} = -abcf$

My answer: B

4. (1 point) Find x if

$$\begin{array}{rcrcl} x & + & (1-i)y & = & -1+2i \\ ix & + & 2y & = & -4 \end{array}$$

A. $x = 1+i$

B. $x = -2+3i$

C. $x = 3-4i$

D. $x = -3+4i$

E. $x = 7-i$

F. $x = 5+2i$

augmented matrix $\begin{bmatrix} 1 & 1-i & -1+2i \\ i & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i & -1+2i \\ 0 & 2-i(1-i) & 4-i(-1+2i) \end{bmatrix}$

$= \begin{bmatrix} 1 & 1-i & -1+2i \\ 0 & 1-i & -2+i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & x \\ 0 & 1-i & -2+i \end{bmatrix}$

where $x = -1+2i-(-2+i) = -1+2+2i-i = 1+i$

My answer: A

5. (1 point) The eigenvalues of the matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ are

A. 1 and 2
B. 2 and 3
C. -1 and -3
D. $\pm i$
E. 3 and -3
F. 2 and 4

$C_A(x) = \det(\lambda I_3 - A) = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = \lambda \begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda-3 \end{vmatrix} + 2 \begin{vmatrix} -1 & \lambda-2 \\ -1 & 0 \end{vmatrix}$

$= \lambda(\lambda-2)(\lambda-3) + 2(\lambda-2) = (\lambda-2)(\lambda(\lambda-3)+2) = (\lambda-2)(\lambda^2-3\lambda+2)$

$= (\lambda-2)(\lambda-2)(\lambda-1)$, so the zeros of $C_A(\lambda)$ are $\lambda=1$ and $\lambda=2$

My answer: A

6. (1 point) The general solution of the system of linear differential equations

$$\begin{aligned} f_1' &= f_1 + f_2 \\ f_2' &= 4f_1 - 2f_2 \end{aligned}$$

is of the form below, where $c, d \in \mathbb{R}$ are arbitrary:

A. $ce^{-3x}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

B. $ce^{-2x}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + de^{3x}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

C. $ce^{-2x}\begin{bmatrix} -4 \\ 1 \end{bmatrix} + de^{3x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

D. $ce^{-3x}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

E. $ce^{-3x}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

F. $ce^{-2x}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + de^{3x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$, $C_A(x) = \begin{vmatrix} \lambda-1 & -1 \\ -4 & \lambda+2 \end{vmatrix} = (\lambda-1)(\lambda+2) - 4 =$

$= \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$,

Eigenvector for $\lambda=-3$: $-3I_2 - A = \begin{bmatrix} -4 & -1 \\ -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}$

so $X_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is an eigenvector for $\lambda=-3$

Eigenvector for $\lambda=2$: $2I_2 - A = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

so $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. By Th. 1 in §2.5 the general solution is of the form

$$c\,e^{-3x}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + d\,e^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

My answer: D

7. (1 point) Let A be a $n \times n$-matrix. Which of the following conditions are equivalent to the statement "$AX = B$ is solvable for every $B \in \mathbb{R}^n$"?

(i) $AX = 0$ is uniquely solvable. True
(ii) $\lambda = 0$ is an eigenvalue of A. False
(iii) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, given by $T(X) = AX$, has a non-zero kernel. False
(iv) The columns of A span $\mathbb{R}^n$. True

A. All of them.
B. (ii), (iii) and (iv).
C. (iii) and (iv) only.
D. (i) and (iv) only.
E. (i), (ii) and (iii).
F. (i), (iii) and (iv).

$AX = B$ is solvable for all $B \in \mathbb{R}^n \iff A$ is invertible.
The Invertible Matrix Theorem says:
A invertible $\iff$ $\lambda = 0$ is NOT an eigenvalue
$\iff$ $AX = 0$ is uniquely solvable
$\iff$ The columns of A span $\mathbb{R}^n$
$\iff$ $T : \mathbb{R}^n \to \mathbb{R}^n$, $X \mapsto AX$ is bijective.

By Cor of Th 5 in §5.2, T is bijective $\iff$ T is one-to-one, and by Th 4 in §5.3, T is one-to-one $\iff$ $\operatorname{Ker} T = 0$.

My answer: D

8. (1 point) Let V be a 5-dimensional vector space. Which of the following statements are true?

(i) Any set of 4 vectors is linearly independent. F
(ii) Every set of 6 vectors contains a basis of V. F
(iii) Every linearly independent set of 5 vectors is a basis of V.
(iv) If U is a subspace of V of dimension 5 then $U = V$.

A. All of them.
B. (ii), (iii) and (iv).
C. (iii) and (iv) only.
D. (i) and (iv) only.
E. (i), (ii) and (iii).
F. (i), (iii) and (iv).

(i) false
(ii) false, since the set may not even span V
(iii) true: Th 4 in §5.2
(iv) true: Th 5 in §5.2

My answer: C

9. (1 point) Which of the following are bases of $\mathbb{R}^4$?

(1) $\{[1\ \ 3\ \ -5\ \ 4], [0\ \ 1\ \ 12\ \ -2], [7\ \ 1\ \ 1\ \ -3]\}$

(2) $\{[1\ \ 2\ \ 3\ \ -4], [0\ \ 2\ \ -3\ \ 17], [0\ \ 0\ \ -1\ \ 8], [0\ \ 0\ \ 0\ \ 6]\}$

(3) $\{[-1\ \ 2\ \ -3\ \ 4], [-1\ \ 2\ \ -3\ \ 4], [0\ \ 2\ \ 5\ \ -8], [-3\ \ 4\ \ 9\ \ 10]\}$

A. All three
B. (1) only
C. (2) only
D. (3) only
E. (2) and (3) only
F. None of them

(1) is not a basis since it has only 3 vectors, but $\dim \mathbb{R}^4 = 4$

(3) is not a basis since the first two vectors are the same, hence linearly dependent

(2) is a basis since

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 2 & -3 & 17 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$ has determinant $1 \cdot 2 \cdot (-1) \cdot 6 = -12$

and is therefore invertible.

My answer: C

10. (1 point) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ -x - 5y \\ 6x \end{bmatrix}$$

My answer: $\begin{bmatrix} 1 & 3 \\ -1 & -5 \\ 6 & 0 \end{bmatrix}$

11. For any real number a consider the linear system

$$\begin{array}{rcrcrcr} x & - & 3a\,y & & & = & 3 \\ & & 3y & + & a\,z & = & 1 \\ x & & & + & 2a\,z & = & 5 \end{array}$$

(a) (4 points) Determine the rank of the coefficient matrix C and the rank of the augmented matrix A of this linear system.

(b) (4 points) Find the conditions on a, so that the system has

(i) infinitely many solutions,

(ii) a unique solution, and

(iii) no solution.

In case (i) write done all solutions.

(a) $A = \begin{bmatrix} 1 & -3a & 0 & 3 \\ 0 & 3 & a & 1 \\ 1 & 0 & 2a & 5 \end{bmatrix} = \left[\begin{array}{c|c} C & \begin{matrix} 3 \\ 1 \\ 5 \end{matrix} \end{array}\right] \sim \begin{bmatrix} 1 & -3a & 0 & 3 \\ 0 & 3 & a & 1 \\ 0 & 3a & 2a & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3a & 0 & 3 \\ 0 & 3 & a & 1 \\ 0 & 0 & 2a-a^2 & 2-a \end{bmatrix} = \begin{bmatrix} 1 & -3a & 0 & 3 \\ 0 & 3 & a & 1 \\ 0 & 0 & a(2-a) & 2-a \end{bmatrix} \Rightarrow \text{rank } A = \begin{cases} 2; & a=2 \\ 3; & a \neq 2 \end{cases}$

$C \sim \cdots \begin{bmatrix} 1 & -3a & 0 \\ 0 & 3 & a \\ 0 & 0 & a(2-a) \end{bmatrix} \Rightarrow \text{rank } C = \begin{cases} 2 & a=0 \text{ or } a=2 \\ 3 & a \neq 0 \text{ and } a \neq 2 \end{cases}$

(b) (i) $\Leftrightarrow$ rank A = rank $C = 2 \Leftrightarrow a = 2$. In this case

$A \sim \begin{bmatrix} 1 & -6 & 0 & 3 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so $\begin{array}{l} x + 4z = 5 \\ 3y + 2z = 1 \end{array}$

The general solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5-4t \\ \frac{1}{3}(1-2t) \\ t \end{bmatrix}$, $t \in \mathbb{R}$ arbitrary

(ii) $\Leftrightarrow$ rank A = rank $C = 3 \Leftrightarrow a \neq 0$ and $a \neq 2$

(iii) no solution $\Leftrightarrow$ rank C < rank $A \Leftrightarrow a = 0$

12. (7 points) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

is $x(x-1)^2$. (You do not need to show this.)

(a) (1 point) Find all eigenvalues of A.

(b) (4 points) For each eigenvalue of A find a basis of the corresponding eigenspace.

(c) (2 points) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Justify your answer.

(a) Since the eigenvalues of A are the roots of the characteristic polynomial, A has eigenvalues 0 and 1

(b) $\underline{\lambda = 0}$:

$0E - A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, thus the eigenspace E_0 consists of the solutions of the linear system $\begin{matrix} x + 2z = 0 \\ y - z = 0 \end{matrix}$,

which are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$ arbitrary. A basis of the eigenspace $E_0(A)$ is therefore $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

$\underline{\lambda = 1}$: $E - A = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The corresponding linear system is $x + z = 0$. Its general solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} =$

$= s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ for $s, t \in \mathbb{R}$ arbitrary. A basis of the eigenspace $E_1(A)$ is therefore $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) Since the basis vectors of $E_0(A)$ and $E_1(A)$ add up to 3, A is diagonalizable, $A = PDP^{-1}$ with

$$P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

13. (7 points) For the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 7 \\ 2 & -2 & 5 & 16 \\ 3 & -3 & 8 & 25 \end{bmatrix}$$

find

(a) (2 points) the reduced row echelon form,
(b) (1 point) a basis of the row space,
(c) (1 point) a basis of the column space,
(d) (2 points) a basis of the null space,
(e) (1 point) the dimension of $\mathrm{col}(A)^\perp$, the orthogonal complement of the column space of A.

(a) $$A \sim \begin{bmatrix} 1 & -1 & 2 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

(b) A basis of row(A) is $\{[1,-1,0,3],[0,0,1,2]\}$, the nonzero rows of R

(c) A basis of col(A) is given by the columns corresponding to the columns with leading 1's in R, i.e. column 1 and column 3:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$

(d) The null space consists of the solutions of the hom. of linear system with coefficient matrix A. Hence they are the solutions of $RX = 0$:

$$x_1 - x_2 + 3x_4 = 0, \quad x_3 + 2x_4 = 0, \quad \text{so} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-3t \\ s \\ -2t \\ t \end{bmatrix} =$$

$$= s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R} \text{ arbitrary.}$$

A basis of null(A) is therefore $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

(e) Since $\mathrm{col}(A) \subset \mathbb{R}^3$, Th. 3 in §4.6 says

$$\dim \mathrm{col}(A)^\perp = 3 - \dim \mathrm{col}(A) = 3 - 2 = 1.$$

14. (a) (3 points) Are $\{p_1 = -1 + x, p_2 = 2 + x^2, p_3 = -7 + 3x - 2x^2\}$ linearly independent in $\mathbb{P}_2$? If no, provide a nontrivial linear combination of p_1, p_2, p_3 that vanishes.

Assume $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$, i.e.

$$0 = c_1(-1+x) + c_2(2+x^2) + c_3(-7+3x-2x^2) =$$
$$= (-c + 2c_2 - 7c_3) + (c_1 + 3c_3)x + (c_2 - 2c_3)x^2,$$

i.e.
$$\begin{aligned} -c_1 + 2c_2 - 7c_3 &= 0 \\ c_1 + 3c_3 &= 0 \\ c_2 - 2c_3 &= 0 \end{aligned} \quad \rightsquigarrow \begin{bmatrix} -1 & 2 & -7 \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \text{ arbitrary.}$$

Since this system has nontrivial solutions $\{p_1, p_2, p_3\}$ are linearly dependent. A nontrivial linear combination that vanishes is

$$-3(-1+x) + 2(2+x^2) + (-7+3x-2x^2) = 0$$

(b) (2 points) Find a spanning set of the subspace $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c = 0 \right\}$ of $\mathbb{M}_{22}$.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{22}$ we have $A \in V \Leftrightarrow A = \begin{bmatrix} a & b \\ -(a+b) & d \end{bmatrix}$

$$\Leftrightarrow \quad A = a \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad a, b, d \in \mathbb{R} \text{ arbitrary}$$

Hence

$$V = \left\{ a \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$$
$$= \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

So $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a spanning set of V.

(c) (3 points) Show that $U = \{f \in \mathbb{F}[0,1] : f(1) = 0\}$ is a subspace of $\mathbb{F}[0,1]$.

We verify the 3 conditions of the subspace test:

- the zero of $\mathbb{F}[0,1]$ is the zero function, $\mathbb{0}: [0,1] \to \mathbb{R} : t \mapsto 0$, mapping every $t \in [0,1]$ to $0 \in \mathbb{R}$, in particular $\mathbb{0}(1) = 0$, so $\mathbb{0} \in U$
- let $f, g \in U$, i.e. $f(1) = 0 = g(1)$. Then $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$, so $(f+g) \in U$
- let $f \in U$, i.e. $f(1) = 0$, and let $s \in \mathbb{R}$. Then $(sf)(1) = s f(1) = s0 = 0$, so $sf \in U$.

15. (a) (3 points) Find an orthogonal basis of the subspace of $\mathbb{R}^4$ spanned by the vectors $X_1 = [1 \ \ 1 \ \ 1 \ \ 1]$, $X_2 = [2 \ \ 4 \ \ 5 \ \ 1]$ and $X_3 = [\ 0 \ \ -4 \ \ -2 \ \ 1]$.

We perform the Gram-Schmidt algorithm:

$F_1 = X_1$

$$F_2 = X_2 - \frac{X_2\cdot F_1}{F_1\cdot F_1}F_1 = [2,4,5,1] - \frac{12}{4}[1,1,1,1] = [2,4,5,1] - [3,3,3,3]$$

$$= [-1,1,2,-2]$$

$$F_3 = X_3 - \frac{X_3\cdot F_1}{F_1\cdot F_1}F_1 - \frac{X_3\cdot F_2}{F_2\cdot F_2}F_2 = [0,-4,-2,1] - \frac{-8}{4}[1,1,1,1] - \frac{0-4-4-2}{10}[-1,1,2,-2]$$

$$= [0,-4,-2,-1] + [2,2,2,2] + [-1,1,2,-2] = [1,-1,2,-1]$$

(for $X_3 = [-3 \ -4 \ -2 \ 1]$ the solution is F_1, F_2 as above, while $F_3 = \frac{1}{10}[-17,-13,14,16]$)

(3 points)

(b) Find the best approximation to a solution of the inconsistent linear system

$$\begin{array}{rcrcl} 4x & & & = & 2 \\ & & 2y & = & 0 \\ x & + & y & = & 11 \end{array}$$

Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$. Since the lin system is (obviously) inconsistent we solve $(A^TA)Z = A^TB$. We have

$$A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}, \quad (A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$A^TB = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$(A^TA)Z = A^TB \iff Z = (A^TA)^{-1}(A^TB) = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 95-11 \\ -19+187 \end{bmatrix}$$

$$= \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad = \text{best approximation.}$$

16. (4 bonus points) Let $T : V \to W$ be a linear transformation, let $\{b_1, \ldots, b_r\}$ be a basis of $\ker T$ and let $\{T(d_1), \ldots, T(d_s)\}$ be a basis of $\operatorname{im} T$, where $d_1, \ldots, d_s$ are suitable vectors in V. Show that $B = \{b_1, \ldots, b_r, d_1, \ldots, d_s\}$ is a basis of V, and conclude that $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$.

see the proof of Th 5 in §5.3