University of Ottawa

Department of Mathematics and Statistics

MAT 1341: Introduction to Linear Algebra

Instructor: Erhard Neher

Final Exam; July 17, 2006

Family Name: ____________________

First Name: ____________________

Student number: ____________________

Please read these instructions carefully:

- You have 180 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.
- The first 9 questions are multiple choice, questions 10 is a short-answer question. Questions 1-10 are worth 1 point each and no part marks will be given. However, you must show some work to obtain the point. Simply writing the correct answer will earn you 0. Please record your answers in the space provided.
- Questions 11-15 are long-answer questions, worth 6-8 points as indicated below. Partial marks may be earned. Please be careful to include all details, and explain precisely what your are doing.
- The last question is a bonus "proof question". You can earn 4 bonus points for a correct proof.
- If you do not have enough space, use the back of the pages and clearly indicate this.
- Where it is possible to check your work, do so.
- Read each question carefully - you will save yourself time and unnecessary grief later on.

Good luck! Bonne chance!

Quest.	1 – 10	11	12	13	14	15	16	Total
maximal	10	8	7	7	8	6	(4 bonus)	46
score								

1. (1 point) Let $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$. If $(2X + A^T)^T = 2A$ then the sum of the entries on the diagonal of X is

A. -1

B. 4

C. 2

D. 0

E. 10

F. 3

My answer:________________

2. (1 point) Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$. The first row of A^{-1} is

A. $\begin{bmatrix} 1 & -4 & -9 \end{bmatrix}$

B. $\begin{bmatrix} 3 & 0 & 4 \end{bmatrix}$

C. $\begin{bmatrix} 2 & -3 & 14 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}$

E. $\begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$

F. $\begin{bmatrix} 1 & -5 & 6 \end{bmatrix}$

My answer:________________

3. (1 point) Calculate the determinant

$$\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 3 \\ c & 0 & 0 & d \\ e & f & 4 & g \end{vmatrix}$$

A. $4acfg$

B. $-abcf$

C. $3abcfg$

D. $-3abcfg$

E. $12abc$

F. $-abcfg$

My answer:________________

4. (1 point) Find x if

$$\begin{array}{ccccc} x & + & (1-i)y & = & -1+2i \\ ix & + & 2y & = & -4 \end{array}$$

A. $x = 1 + i$

B. $x = -2 + 3i$

C. $x = 3 - 4i$

D. $x = -3 + 4i$

E. $x = 7 - i$

F. $x = 5 + 2i$

My answer:________________

5. (1 point) The eigenvalues of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ are

A. 1 and 2

B. 2 and 3

C. -1 and -3

D. $\pm i$

E. 3 and -3

F. 2 and 4

My answer:________________

6. (1 point) The general solution of the system of linear differential equations

$$\begin{aligned} f_1' &= f_1 + f_2 \\ f_2' &= 4f_1 - 2f_2 \end{aligned}$$

is of the form below, where c, $d \in \mathbb{R}$ are arbitrary:

A. $ce^{-3x}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

B. $ce^{-2x}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + de^{3x}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

C. $ce^{-2x}\begin{bmatrix} -4 \\ 1 \end{bmatrix} + de^{3x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

D. $ce^{-3x}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

E. $ce^{-3x}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

F. $ce^{-2x}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + de^{3x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

My answer:________________

7. (1 point) Let A be a $n \times n$-matrix. Which of the following conditions are equivalent to the statement "$AX = B$ is solvable for every $B \in \mathbb{R}^n$".

(i) $AX = 0$ is uniquely solvable.
(ii) $\lambda = 0$ is an eigenvalue of A.
(iii) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, given by $T(X) = AX$, has a non-zero kernel.
(iv) The columns of A span $\mathbb{R}^n$.

A. All of them.

B. (ii), (iii) and (iv).

C. (iii) and (iv) only.

D. (i) and (iv) only.

E. (i), (ii) and (iii).

F. (i), (iii) and (iv).

My answer:_______________

8. (1 point) Let V be a 5-dimensional vector space. Which of the following statements are true?

(i) Any set of 4 vectors is linearly independent.
(ii) Every set of 6 vectors contains a basis of V.
(iii) Every linearly independent set of 5 vectors is a basis of V.
(iv) If U is a subspace of V of dimension 5 then $U = V$.

A. All of them.

B. (ii), (iii) and (iv).

C. (iii) and (iv) only.

D. (i) and (iv) only.

E. (i), (ii) and (iii).

F. (i), (iii) and (iv).

My answer:_______________

9. (1 point) Which of the following are bases of $\mathbb{R}^4$?

(1) $\{[1 \quad 3 \quad -5 \quad 4], [0 \quad 1 \quad 12 \quad -2], [7 \quad 1 \quad 1 \quad -3]\}$

(2) $\{[1 \quad 2 \quad 3 \quad -4], [0 \quad 2 \quad -3 \quad 17], [0 \quad 0 \quad -1 \quad 8], [0 \quad 0 \quad 0 \quad 6]\}$

(3) $\{[-1 \quad 2 \quad -3 \quad 4], [-1 \quad 2 \quad -3 \quad 4], [0 \quad 2 \quad 5 \quad -8], [-3 \quad 4 \quad 9 \quad 10]\}$

A. All three

B. (1) only

C. (2) only

D. (3) only

E. (2) and (3) only

F. None of them

My answer:________________

10. (1 point) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ -x - 5y \\ 6x \end{bmatrix}$$

My answer:________________________________

11. For any real number a consider the linear system

$$\begin{array}{rcrcrcl} x & - & 3a\,y & & & = & 3 \\ & & 3y & + & a\,z & = & 1 \\ x & & & + & 2a\,z & = & 5 \end{array}$$

(a) (4 points) Determine the rank of the coefficient matrix C and the rank of the augmented matrix A of this linear system.

(b) (4 points) Find the conditions on a, so that the system has

(i) infinitely many solutions,
(ii) a unique solution, and
(iii) no solution.

In case (i) write done all solutions.

12. (7 points) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

is $x(x-1)^2$. (You do not need to show this.)

(a) (1 point) Find all eigenvalues of A.

(b) (4 points) For each eigenvalue of A find a basis of the corresponding eigenspace.

(c) (2 points) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Justify your answer.

13. (7 points) For the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 7 \\ 2 & -2 & 5 & 16 \\ 3 & -3 & 8 & 25 \end{bmatrix}$$

find

(a) (2 points) the reduced row echelon form,
(b) (1 point) a basis of the row space,
(c) (1 point) a basis of the column space,
(d) (2 points) a basis of the null space,
(e) (1 point) the dimension of $\operatorname{col}(A)^{\perp}$, the orthogonal complement of the column space of A.

14. (a) (3 points) Let $p_1 = -1 + x$, $p_2 = 2 + x^2$, $p_3 = -7 + 3x - 2x^2$. Is $\{p_1, p_2, p_3\}$ linearly independent in $\mathbb{P}_2$? If no, provide a nontrivial linear combination of p_1, p_2, p_3 that vanishes.

(b) (2 points) Find a spanning set of the subspace $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c = 0 \right\}$ of $\mathbb{M}_{22}$.

(c) (3 points) Show that $U = \{f \in \mathbb{F}[0,1] : f(1) = 0\}$ is a subspace of $\mathbb{F}[0,1]$.

15. (a) (3 points) Find an orthogonal basis of the subspace of $\mathbb{R}^4$ spanned by the vectors $X_1 = [\,1 \quad 1 \quad 1 \quad 1\,]$, $X_2 = [\,2 \quad 4 \quad 5 \quad 1\,]$ and $X_3 = [\,0 \quad -4 \quad -2 \quad 1\,]$.

(b) (3 points) Find the best approximation to a solution of the inconsistent linear system

$$\begin{array}{rcrcr} 4x & & & = & 2 \\ & & 2y & = & 0 \\ x & + & y & = & 11 \end{array}$$

16. (4 bonus points) Let $T : V \to W$ be a linear transformation, let $\{b_1, \ldots, b_r\}$ be a basis of $\ker T$ and let $\{T(d_1), \ldots, T(d_s)\}$ be a basis of $\operatorname{im} T$, where $d_1, \ldots, d_s$ are suitable vectors in V. Show that $B = \{b_1, \ldots, b_r, d_1, \ldots, d_s\}$ is a basis of V, and conclude that $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$.