1. The area of the triangle with vertices $A(1,1,1)$, $B(0,1,2)$ and $C(1,0,2)$ is

A. $\frac{1}{2}\sqrt{2}$
B. $\frac{1}{2}\sqrt{3}$
C. $\sqrt{2}$
D. $\frac{3}{2}$
E. 5
F. 2

(see §3.5, Example 2)

$\overrightarrow{AB} = [0\ 1\ 2]^T - [1\ 1\ 1]^T = [-1\ 0\ 1]^T$

$\overrightarrow{AC} = [1\ 0\ 2]^T - [1\ 1\ 1]^T = [0\ -1\ 1]^T$

My answer: B

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} =$$

$$= \left[\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}, -\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \right]^T = [1\ 1\ 1]^T$$

$$\text{area} = \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \frac{1}{2}\sqrt{1+1+1} = \frac{1}{2}\sqrt{3}$$

2. The shortest distance from the point $P(2,7,9)$ to the line through $P_0(-1,-2,0)$ with direction vector $\vec{d} = [-1\ \ 2\ \ 5]^T$ is

A. $\sqrt{51}$
B. $\sqrt{37}$
C. 3
D. $\sqrt{20}$
E. 0
F. 4

(see §2.3 Example 9)

$\vec{v} = \overrightarrow{P_0P} = [2\ 7\ 9]^T - [-1\ -2\ 0]^T = [3\ 9\ 9]^T$

$$\vec{v}_1 = \text{proj}_{\vec{v}}(\vec{d}) = \frac{\vec{v}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{[3\ 9\ 9]\cdot[-1\ 2\ 5]^T}{1+4+25}\vec{d}$$

My answer: A

$$= \frac{1}{30}(-3+18+45)\vec{d} = 2\vec{d} =$$

$$= [-2\ 4\ 10]$$

$$\|\vec{v} - \vec{v}_1\| = \| [3\ 9\ 9]^T - [-2\ 4\ 10]^T \| = \| [5\ 5\ 1]^T \| =$$

$$= \sqrt{25+25+1} = \sqrt{51}$$

5. The vector equation of the line through $P(8,4,3)$ and parallel to the line through $A(4,-1,2)$ and $B(-1,3,0)$ is

A. $[1\ \ 0\ \ 3]^T + t[7\ \ 4\ \ 0]^T$
B. $[3\ \ 0\ \ 2]^T + t[5\ \ 9\ \ -2]^T$
C. $[1\ \ 1\ \ 1]^T + t[0\ \ 3\ \ 7]^T$
D. $[6\ \ 2\ \ 1]^T + t[2\ \ 2\ \ 2]^T$
E. $[13\ \ 0\ \ 5]^T + t[5\ \ -4\ \ 2]^T$
F. $[1\ \ 0\ \ 3]^T + t[5\ \ 9\ \ 2]^T$

A direction vector of the line through A and B

$\vec{d} = [4\ -1\ 2]^T - [-1\ 3\ 0]^T = [5\ -4\ 2]^T$,

Thus the line has vector equation

$[8\ 4\ 3]^T + t[5\ -4\ 2]^T$

My answer: E

For $t = 1$ we see that the point $[8\ 4\ 3]^T + [5\ -4\ 2]^T$

$= [13\ 0\ 5]^T$ lies on the line. Hence also

$[13\ 0\ 5]^T + t[5\ -4\ 2]^T$ is a vector equation for the line

6. The volume of the parallelepiped defined by the three vectors $\vec{u} = [1\ \ -1\ \ 0]^T$, $\vec{v} = [2\ \ 1\ \ 3]^T$ and $\vec{w} = [4\ \ 1\ \ -1]^T$ is

A. -9
B. 9
C. 4
D. 18
E. 5
F. 3

(see §3.5 Th. 6)

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} + 4\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix}$$

$= (-1-3) - 2(1) + 4(-3) =$

$= -4 - 2 - 12 = -18$

My answer: D

volume $= |\vec{u}\cdot(\vec{v}\times\vec{w})| = 18$

3. The equation of the plane through $P(2,1,-1)$ and perpendicular to the line through $Q(4,3,-2)$ and $R(2,1,0)$ is

A. $x+y-z=4$
B. $2x+y-z=6$
C. $3x-y-z=6$
D. $2x+y-z=0$
E. $4x+3y-2z=1$
F. $2x+y=5$

A direction vector of the line is $\vec{d} = [2\ 1\ 0]^T - [4\ 3\ -2]^T$
$= [-2, -2, 2]^T = 2\,[-1\ -1\ 1]^T$.
The plane is perpendicular to the given line
$\iff \vec{d}$ is a normal of the plane

My answer: A

Hence an equation of the plane is $-2x-2y+2z=d$, where d can be determined by the condition that $P(2,1,-1)$ lies on the plane: $-2(2)-2(1)+2(-1) = -4-2-2 = -8 = d$, so $-2x-2y+2z=-8$, equivalently $x+y-z=4$

4. The equation of the plane through $P(1,1,0)$ and parallel to the plane through $A(3,-1,0)$ and $B(4,1,1)$ and $C(0,1,0)$ is

A. $7x+4y=11$
B. $3x-y+4z=2$
C. $4x+y+z=5$
D. $-3x+2y-4z=-1$
E. $2x+3y-8z=5$
F. $4x-4y+2z=0$

(Compare §3.3, Example (2))
We determine an equation for the plane through A, B and C, starting from the general form of such an equation, say $ax+by+cz=d$.
Substituting the coordinates

My answer: E

of A, B, C in $ax+by+cz=d$ yields the 3 equations
$3a-b=d$, $4a+b+c=d$, $b=d$, whence $3a=2b=2d$, $a=\frac{2}{3}d$,
$c=-4a-b+d=-4a=-\frac{8}{3}d$. An equation for the plane through A, B, C is therefore (take $d=3$): $2x+3y-8z=3$. Thus this plane has normal $[2\ 3\ -8]^T$. The plane we are looking for therefore is $2x+3y-8z=e$, where e is determined by the condition that $P(1,1,0)$ lies on the plane: $2+3=e=5$.
Thus $2x+3y-8z=5$

7. Find the vector equation of the line through $P(1,0,3)$, intersecting the line $[x \quad y \quad z]^T = [3 \quad 4 \quad 0]^T + t[1 \quad -1 \quad 0]^T$ and perpendicular to that line.

A. $[1 \quad 0 \quad 3]^T + t[2 \quad 2 \quad -5]^T$

B. $[1 \quad 0 \quad 3]^T + t[1 \quad 1 \quad -1]^T$

C. $[3 \quad 0 \quad 4]^T + t[7 \quad 3 \quad 2]^T$

D. $[-10 \quad -7 \quad 1]^T + t[11 \quad 7 \quad -4]^T$

E. There is no such line

F. $[3 \quad 4 \quad 0]^T + t[-2 \quad -5 \quad 3]^T$

My answer: B

An arbitrary point on the given line has the form $[3+t, 4-t, 0]^T$. Any line containing P and such a point has direction vector $[3+t, 4-t, 0]^T - [1\ 0\ 3]^T = [2+t, 4-t, -3]^T$. Such a direction vector is orthogonal to the given line $\iff 0 = [2+t, 4-t, -3]^T \cdot [1, -1, 0]^T = 2+t-(4-t) = -2+2t$ $\iff t = 1$. Thus the line we are looking for has direction vector $\vec{d} = [3, 3, -3]^T$, and also $[1, 1, -1]^T$. It can be parametrized by $[1\ 0\ 3]^T + t[1\ 1\ -1]^T$

8. The roots of the quadratic polynomial $x^2 - 8x + 18$ are

A. $7 \pm \sqrt{5}\,i$

B. $-2 \pm \sqrt{5}\,i$

C. $5 \pm \sqrt{3}\,i$

D. $-2 \pm \sqrt{3}\,i$

E. $-3 \pm i$

F. $4 \pm \sqrt{2}\,i$

$x^2 - 8x + 18 = 0 \iff x^2 - 8x = -18 \iff$

$(x-4)^2 = -18 + 16 = -2$

$\iff x - 4 = \pm\sqrt{2}\,i$

$\iff x = 4 \pm \sqrt{2}\,i$

My answer: F

9. The complex number z satisfying $(2+i)z+(3-i)=(3-2i)z+(1-2i)$ is

A. $\frac{1}{10}(-1+7i)$

B. $\frac{1}{5}(-2-i)$

C. $\frac{1}{5}(2+3i)$

D. $\frac{1}{10}(-1-i)$

E. $\frac{1}{5}(1+i)$

F. $\frac{1}{5}(-2-3i)$

My answer: A

$$(2+i)z-(3-2i)z = 1-2i-(3-i)$$

$$(2+i-3+2i)z = -2-2i+i$$

$$(-1+3i)z = -2-i$$

$$z = \frac{-2-i}{-1+3i} = \frac{(-2-i)(-1-3i)}{(-1+3i)(-1-3i)}$$

$$= \frac{(-2)(-1)+(-2)(-3i)+i+3i^2}{1+9} = \frac{1}{10}(2+6i+i-3)$$

$$= \frac{1}{10}(-1+7i)$$