University of Ottawa

Department of Mathematics and Statistics

MAT 1341 B: Introduction to Linear Algebra

Instructor: Erhard Neher

Test 3; Nov. 22, 2007, 17:30-18:50

Family Name: ______________________________

First Name: ______________________________

Student number: ______________________________

The last digit of your student number is $\alpha =$

The second last digit of your student number is $\beta =$

Please read these instructions carefully:

- Enter your name on this page and the next, but your student number only on this page. You will get back the test without this first page.
- The table below is for the TA. Do not write in the table. For privacy reasons, this page of the test will be detached, and you will only get back the remaining pages of the test. Therefore, **fill in your name on both pages** and your student number on this page only.
- No books or notes are allowed. **Calculators are not permitted.**

Good luck!

Quest.	1	2	3	4	5	Total
maximal	6	6	$3 + 3$ bonus	8	6	29
score						

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Family Name: ______________________________

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Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
- For all questions you must show your work to obtain the points. Simply writing the correct answer will earn you 0.
- Please write legibly and argue logically: You must convince the TA that you know why your solution is correct.
- No books or notes are allowed. **Calculators are not permitted.**

1. (6 points) Let α denote the LAST digit of your student number. For each of the following three sets of vectors, answer the three questions:
(a) Is the set linearly independent?
(b) Do the vectors span $\mathbb{R}^3$?
You must correctly justify your answer to obtain any marks. (This question has two pages.)
(i)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ \alpha - 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Answer: (a) Since the set has two vectors and they are not parallel, they are linearly independent, YES.

(b) Since $\mathbb{R}^3$ is 3-dimensional, any spanning set must have at least 3 vectors. This set has only two vectors, so NO.

(ii)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ \alpha - 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Answer: Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 0 \\ \alpha - 9 & 1 & -1 \end{bmatrix}.$$

We know that the columns of A will be linearly independent and will span $\mathbb{R}^3$ if and only if A is invertible. To decide this, we could row reduce A to see if its RREF would have 3 leading ones; or simply take the determinant:

$$\det(A) = \det \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 0 \\ \alpha - 9 & 1 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 0 \\ \alpha - 8 & 3 & 0 \end{bmatrix} = 1(-3 + 2(\alpha - 8)) = 2\alpha - 19 \neq 0$$

(where we added R1 to R3 in the second step to simplify the calculation of the determinant). Thus this matrix is invertible and so the answer to both questions is YES.

(iii)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\}$$

Answer: (a) Since $\dim(\mathbb{R}^3) = 3$, any set with more than 3 vectors is linearly dependent. We have four vectors so NO.

(b) We row reduce the matrix whose columns are these four vectors

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, we deduce that not every linear system with A as a coefficient matrix will be consistent, which implies that not every vector in $\mathbb{R}^3$ is a linear combination of our given 4 vectors. It follows that these four vectors do NOT span $\mathbb{R}^3$.

2. (6 points) For each of the sets given below:
(a) Give an example of an element in the set.
(b) Is it a subspace of $\mathbb{M}_{2,2}$? Justify your answer.

(i)

$$U = \{A \in \mathbb{M}_{2,2} : A = A^T\}$$

Answer: (a) The identity matrix equals its own transpose, so I is in U.

(b) Yes, it is. We verify this with the subspace test.

(1): The zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which equals its transpose; so the zero matrix is in U.

(2): If A and B are two matrices for which $A = A^T$ and $B = B^T$, then consider $A + B$. This sum is in U if and only if it equals its transpose, so we calculate:

$$(A + B)^T = A^T + B^T = A + B$$

where the first equality comes from properties of the transpose and the second is our hypotheses on A and B. So $A + B$ is in U.

(3): If $A^T = A$ and c is a scalar, then $(cA)^T = cA^T$ by properties of the transpose and $cA^T = cA$ since $A^T = A$. So $(cA)^T = cA$ which means cA is in U.

Thus U is a subspace of $\mathbb{M}_{2,2}$.

(ii)

$$W = \{A : \det(A) = 1\}$$

Answer: (a) The identity matrix has determinant 1, so lies in W.

(b) No, this isn't a subspace, because the zero matrix has determinant zero, not 1. So 0 is not in W, so W fails the subspace test.

3. (a) (3 points) Prove that the set

$$\{1 + x - x^2,\, 2 - x,\, 1 + x^3\}$$

is a linearly independent subset of $\mathbb{P}_3$.

Answer: We need to show that the only solution to the equation

$$c_1(1 + x - x^2) + c_2(2 - x) + c_3(1 + x^3) = 0$$

is $c_1 = c_2 = c_3 = 0$. So we expand the left side and gather like terms:

$$(c_1 + 2c_2 + c_3) + (c_1 - c_2)x + (-c_1)x^2 + c_3x^3 = 0$$

This equation can hold if and only if

$$c_1 + 2c_2 + c_3 = 0, \;\; c_1 - c_2 = 0 \;\;\; -c_1 = 0 \;\; c_3 = 0$$

The last two equations force $c_1 = c_3 = 0$ and then the third forces $c_2 = 0$. Thus the only solution is the trivial one, implying these polynomials are linearly independent.

(b) (**3 bonus points**) Prove that the set

$$\{\frac{1}{x-2}, \frac{1}{x-3}, \frac{1}{(x-2)(x-3)}\}$$

is a linearly dependent subset of the space of continuous functions with domain $[0, 1]$.

Answer: We try to solve the equation

$$c_1\frac{1}{x-2} + c_2\frac{1}{x-3} + c_3\frac{1}{(x-2)(x-3)} = 0$$

Multiplying through by $(x - 2)(x - 3)$ gives

$$c_1(x - 3) + c_2(x - 2) + c_3 = 0$$

Gathering like terms we deduce

$$(-3c_1 - 2c_2 + c_3) + (c_1 + c_2)x = 0$$

so necessarily

$$-3c_1 - 2c_2 + c_3 = 0 \;\; c_1 + c_2 = 0$$

But this is a homogeneous linear system with 3 variables and 2 equations and therefore has infinitely many solutions. We conclude that there exists a nontrivial solution and in fact the functions are linearly dependent.

4. (8 points) Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & -1 & 0 & 1 \\ -3 & 2 & 1 & -2 \\ 4 & 1 & 6 & 1 \end{bmatrix}$$

(a) (4 points) Find the reduced row-echelon form of A.

(b) (1 points) Find a basis of the row space of A and its dimension.

(c) (1 points) Find a basis of the column space of A and its dimension.

(d) (2 points) Find a basis of the null space of A and its dimension.

Answer: We perform elementary row operations:

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & -1 & 0 & 1 \\ -3 & 2 & 1 & -2 \\ 4 & 1 & 6 & 1 \end{bmatrix} \overset{(1)}{\sim} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 5 & 10 & 1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \overset{(2)}{\sim} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 1 \\ 0 & -3 & -6 & -1 \end{bmatrix}$$

$$\overset{(3)}{\sim} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{(4)}{\sim} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{(5)}{\sim} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{(6)}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: R$$

Explanations: (1) subtract multiples of the 1st row from the rows below; (2) divide row 4 by -3 and exchange row 4 and row 2; (3) subtract multiples of row 2 from the rows below; (4) divide row 3 by -4 and subtract 2 row 3 from row 4; (5) subtract multiples of row 3 from the 2 rows above; subtract row 2 from row 1. The last matrix is the reduced row-echelon form of A.

(b) By Theorem 1 on page 204, the non-zero rows of R form a basis of the row space of A. Thus, a basis of $row(A)$ is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

and its dimension is $3 = \text{rank}\,(A)$.

(b) According to Theorem 2 on page 205, the columns of A corresponding to the columns of R with leading 1's are a basis of A. These are the columns 1, 2 and 4. Thus a basis of the columns space $colA$ is

$$\begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

and its dimension is also 3, in accordance with Theorem 2.

(c) The null space is the same as the space of solutions of the homogeneous linear system $AX = 0$, which is also the space of solutions of the system $X = 0$:

$$x_1 + x_3 = 0; \quad x_2 + 2x_3 = 0; \quad x_4 = 0$$

The general solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s \\ -2s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

By Theorem 6 on page 209, a basis of the null space of A is given by the basic solutions, i.e., by

$$\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Hence, the dimension of the null space is 1. Since the rank of A is 3 and the system has 4 variables, this confirms Theorem 6 ($1 = 4 - 3$).

5. (6 points) Let β denote the second-last digit of your student number. Find a basis for the subspace of $\mathbb{P}_2$ given by

$$U = \{p(x) \in \mathbb{P}_2 : p(9-\beta) = 0\}$$

and determine $\dim(U)$.

Answer: Let us abbreviate $\rho = 9 - \beta$. An arbitrary polynomial in $\mathbb{P}_2$ has the form

$$p(x) = a_0 + a_1 x + a_2 x^2, \quad (a_i \in \mathbb{R}).$$

For such a polynomial

$$p(\rho) = 0 \iff a_0 + a_1\rho + a_2\rho^2 = 0 \iff a_0 = -(a_1\rho + a_2\rho^2),$$

and therefore

$$p \in U \iff p(x) = -(a_1\rho + a_2\rho^2) + a_1 x + a_2 x^2 \iff p(x) = a_1(x-\rho) + a_2(x^2 - \rho^2).$$

Hence

$$U = span\,\{x - \rho, x^2 - \rho^2\}$$

To prove that $\{x-\rho, x^2-\rho^2\}$ is a basis of U, it remains to show that the set is linearly independent. Suppose therefore that $a_1(x-\rho) + a_2(x^2-\rho^2) = 0$. Collecting terms, we get

$$-(a_1\rho + a_2\rho^2) + a_1 x + a_2 x^2 = 0$$

and then by comparison of coefficients that $a_1 = 0 = a_2$. Thus, $\{x-\rho, x^2-\rho^2\}$ is a basis of U and $\dim U = 2$ follows.