University of Ottawa

Department of Mathematics and Statistics

MAT 1341 B: Introduction to Linear Algebra

Instructor: Erhard Neher

Test 2; Oct. 25, 2007, 17:30-18:50

Family Name: ______________________________

First Name: ______________________________

Student number: ___________________________

The last digit of your student number is $\alpha =$

The third last digit of your student number is $\gamma =$

Please read these instructions carefully:

- Enter your name on this page and the next, but your student number only on this page. You will get back the test without this first page.
- The table below is for the TA. Do not write in the table. For privacy reasons, this page of the test will be detached, and you will only get back the remaining pages of the test. Therefore, **fill in your name on both pages** and your student number on this page only.
- No books or notes are allowed. **Calculators are not permitted.**

Good luck!

Quest.	$1-6$	7	8	9	10	Total
maximal	12	3	6	6	2(bonus)	27
score						

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Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
- Questions 1-5 are multiple choice questions. No part marks will be given. Questions 6-9 require a detailed answer; you may earn part marks for these. Question 10 is a bonus question.
- Please write legibly and argue logically: You must convince the TA that you know why your solution is correct.
- No books or notes are allowed. **Calculators are not permitted.**

1. (2 points) Let A be an $m \times n$ matrix with $m \neq n$. Which of the following statements are true or false for the corresponding homogeneous linear system $AX = 0$:

(i) *If* $\operatorname{rank}(A) = r$*, the system has exactly* $m - r$ *basic solutions.*
(ii) *If* $m > n$*, the system has a nontrivial solution.*
(iii) *If* $\operatorname{rank}(A) = m$*, the system has only the trivial solution.*

A. true, false, true
B. false, true, false
C. true, true, false
D. true, false, false
E. false, false, true
F. false, false, false

Answer: (i) The number of basic solutions is $n - r$, and not $m - r$. Hence (i) is false. (ii) If $m > n$, the system has more equations then variables. It could still be the case that the system has only the trivial solution, i.e., that $n = r$. For example, the system $x = 0$, $2x = 0$ has $m = 2$, $n = r = 1$. So (ii) is false. (iii) If $\operatorname{rank} A = m$, then $m < n$ since the rank of A is always at most m or n and since $m \neq n$. Hence $n - r > 0$ and the system has a nontrivial solution. So (iii) is also false, and (F) is the correct answer.

My answer:_______________

2. (2 points) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ be two 3×4 matrices and let $A^T B = \begin{bmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$.

Which of the following formulas is correct?

A. $c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} + a_{24}b_{43}$
B. $c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$
C. $c_{23} = a_{21}b_{31} + a_{22}b_{32} + a_{23}b_{33}$
D. $c_{23} = a_{21}b_{31} + a_{22}b_{32} + a_{23}b_{33} + a_{24}b_{34}$
E. $c_{23} = a_{12}b_{13} + a_{22}b_{23} + a_{32}b_{33}$
F. $c_{23} = a_{12}b_{13} + a_{22}b_{23} + a_{32}b_{33} + a_{42}b_{43}$

Answer: c_{23} is the (23)-entry of the matrix $A^T B$, which is the dot product of the second row of A^T and the third column of B. Since the second row of A^T is the second column of A, we get

$$c_{23} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \cdot \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = a_{12}b_{13} + a_{22}b_{23} + a_{32}b_{33}$$

Hence (E) is correct.

My answer:_______________

3. (2 points) Let A and B be $n \times n$ matrices. Among the following statements, which one is **false**?

A. If there exists an $n \times n$-matrix such that $AC = I_n$, then A is invertible.
B. If A and B are invertible, then AB is invertible.
C. If A is invertible, then so is A^T.
D. If A is invertible, the linear system $AX = C$ is solvable for every column vector $C \in \mathbb{R}^n$.
E. If A and B are invertible, then $(BA^{-1})^{-1}(AB^{-1})^{-1}A = A$.
F. If A and B are invertible, then $B^{-1}(AB^{-1}A^{-1})^{-1}A = A$.

Answer: A is correct, see the Invertible Matrix Theorem (Corollary 1 of Theorem 5 on page 46). B is correct, see Theorem 3 on page 43. C is correct, see Theorem 3 on page 43. D is correct, see the Invertible Matrix Theorem (Theorem 5 on page 46). E. We use the rules $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^{-1})^{-1} = A$ for invertible matrices. We then get

$$B^{-1}(AB^{-1}A^{-1})^{-1}A = (A^{-1})^{-1}B^{-1}(B^{-1})^{-1}A^{-1}A = AB^{-1}B = A.$$

So E is correct. For F we get

$$B^{-1}(AB^{-1}A^{-1})^{-1} = B^{-1}(A^{-1})^{-1}(B^{-1})^{-1}A^{-1}A = B^{-1}AB \neq A.$$

so F is not correct, and the correct answer is F.

My answer:________________

4. (2 points) Let B be the matrix obtained from the matrix A by performing the following elementary row operations:

(i) exchange the first and fourth row of A,
(ii) add 3 times the second row to the third row,
(iii) multiply the fourth row by 3,
(iv) add -2 times the first row to the the second row,
(v) add $\frac{1}{2}$ times the first row to the fifth row.

Then $\det(B)$ and $\det(A)$ are related as follows:

A. $\det(B) = -3\det(A)$
B. $\det(A) = -3\det(B)$
C. $\det(B) = 9\det(A)$
D. $\det(A) = 9\det(B)$
E. $\det(B) = \det(A)$
F. $\det(A) = -\det(B)$

Answer: Let B_x, $x = (i) - (v)$, be the matrix obtained after the operation x. To calculate the determinant we apply Theorem 2 on page 77 repeatedly: $\det(A) = -\det B_{(i)} = -\det B_{(ii)} = -1/3 \det B_{(iii)} = -1/3 \det B_{(iv)} = -1/3 \det B_{(v)} = -1/3 \det B$. Hence A is correct.

My answer:________________

5. (2 points) The complex eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$ are

A. $\lambda_1 = 1 + \frac{5}{2}i$ and $\lambda_2 = 1 - \frac{5}{2}i$
B $\lambda_1 = 1 + \frac{3}{2}i$ and $\lambda_2 = 1 - \frac{3}{2}i$
C. $\lambda_1 = 2 + 5i$ and $\lambda_2 = 2 - 5i$
D. $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$
E. $\lambda = 2 + 3i$
F. $\lambda = 1 + \frac{5}{2}i$

Answer: We calculate the characteristic polynomial

$$c_A(x) = \begin{vmatrix} x-1 & -2 \\ 5 & x-3 \end{vmatrix} = (x-1)(x-3) + 10 = x - 4x + 13 = (x-2)^2 + 9.$$

Hence the eigenvalues (= the roots of $c_A(x)$) are $2 \pm 3i$. Answer D is correct.

My answer:________________

6. (2 points) Suppose that a directed graph with four vertices has adjacency matrix given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

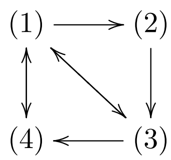
How many paths of length exactly 2 go from vertex 1 to vertex 3? Show your work.

Answer: By Theorem 5 in section 1.4, the number of paths of length 2 from vertex 1 to vertex 3 is the (13)-entry of the matrix A^2. This is obtained as the dot product of the first row of A and the third column of A:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 + 1 + 0 + 0 = 1$$

Thus, the answer is 1.

Second solution: One can draw the graph, and see by inspection that there is exactly 1 path of length 2 from vertex 1 to vertex 3, namely $(1) \to (2) \to (3)$:



My answer:________________

7. Suppose we have two population of competing species, whose growth satisfies the dynamical system $V_{k+1} = AV_k$ where V_k is a 2-vector representing the populations of the two species in year k and

$$A = \begin{bmatrix} -\frac{1}{2} & 3 \\ -\frac{1}{2} & 2 \end{bmatrix}.$$

Suppose we start with an initial population of $V_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. The matrix A has eigenvalues 1 and $\frac{1}{2}$ with associated eigenvectors $X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, respectively.

(a) (2 points) Give a formula for the population V_k at year k as a linear combination of X_1 and X_2.

(b) (1 point) Estimate the long-term population (as $k \to \infty$), if possible.

Answer: (a) The matrix A is diagonalized by the matrix $P = [X_1|X_2]$. Thus

$$P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \qquad P^{-1} = -\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix},$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}V_0 = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 200 \\ -100 \end{bmatrix}.$$

It now follows from Theorem 1 of section 2.6 that

$$V_k = b_1X_1 + b_2(\tfrac{1}{2})^k X_2 = 200\, X_1 - 100\, (\tfrac{1}{2})^k\, X_2$$

It was not required to derive this formula, but here is the proof:

$$\begin{aligned} V_k = A^k V_0 &= (P\,\mathrm{diag}(\lambda_1, \lambda_2)\, P^{-1})^k\, V_0 = P\,\mathrm{diag}(1, (\tfrac{1}{2})^k)\, P^{-1}V_0 \\ &= [X_1|X_2] \begin{bmatrix} b_1 \\ (\frac{1}{2})^k b_2 \end{bmatrix} = b_1X_1 + b_2(\tfrac{1}{2})^k X_2 \end{aligned}$$

(b) As k goes to ∞, $(\frac{1}{2})^k$ goes to 0 and only the term

$$200X_1 = \begin{bmatrix} 400 \\ 200 \end{bmatrix}$$

remains. Hence $200X_1$ is the long term population.

8. (a) (5 points) In the matrix below, **replace α with the last digit of your student number** and find all eigenvalues of A. Hint: One of the eigenvalues is -1.

$$A = \begin{bmatrix} 2\alpha - 7 & -4 & 18 - 2\alpha \\ 3\alpha + 3 & \alpha & -6 - 6\alpha \\ \alpha - 3 & -2 & 8 - \alpha \end{bmatrix}$$

(1 point) Without computing basic eigenvectors, can you decide if A is diagonalizable or not?

Answer: We calculate the characteristic polynomial:

$$c_A(x) = \det(xI_3 - A) = \begin{vmatrix} x - 2\alpha + 7 & 4 & 2\alpha - 18 \\ -3(\alpha+1) & x - \alpha & 6(\alpha+1) \\ 3 - \alpha & 2 & x + \alpha - 8 \end{vmatrix} \overset{(1)}{=} \begin{vmatrix} x + 1 & 0 & -2x - 2 \\ -3(\alpha+1) & x - \alpha & 6(\alpha+1) \\ 3 - \alpha & 2 & x + \alpha - 8 \end{vmatrix}$$

$$\overset{(2)}{=} (x+1) \begin{vmatrix} 1 & 0 & -2 \\ -3(\alpha+1) & x - \alpha & 6(\alpha+1) \\ 3 - \alpha & 2 & x + \alpha - 8 \end{vmatrix}$$

$$\overset{(3)}{=} (x+1) \begin{vmatrix} 1 & 0 & 0 \\ -3(\alpha+1) & x - \alpha & 0 \\ 3 - \alpha & 2 & x - \alpha - 2 \end{vmatrix}$$

$$\overset{(4)}{=} (x+1) \begin{vmatrix} x - \alpha & 0 \\ 2 & x - \alpha - 2 \end{vmatrix} = (x+1)(x-\alpha)(x - (\alpha+2)\,).$$

Explanations: (1) replace row R1 by R1 - 2R3; (2) pull out a factor $x+1$ from row R1; (3) replace column C3 by C3+2C1; (4) expansion along the first row.

Hence, the eigenvalues are -1, α and $\alpha + 2$.

(b) Since the eigenvalues, which are the roots of c_A, are all distinct, A is diagonalizable.

9. In the matrix A below, **replace γ with the third last digit of your student number.** The matrix

$$A = \begin{bmatrix} -\gamma-4 & 0 & -2\gamma-4 \\ \gamma+2 & \gamma & \gamma+2 \\ \gamma+2 & 0 & 2\gamma+2 \end{bmatrix}$$

has eigenvalues γ and -2.

(a) (4 points) For each eigenvalue of A find basic eigenvectors.

(b) (2 points) Decide if A is diagonalizable or not. If yes, give a matrix P and a diagonal matrix D satisfying $P^{-1}AP = D$. If no, explain why A is not diagonalizable.

Answer: For the eigenvalue γ we get

$$\gamma I_3 - A = \begin{bmatrix} 2\gamma+4 & 0 & 2\gamma+4 \\ -\gamma-2 & 0 & -\gamma-2 \\ -\gamma-2 & 0 & -\gamma-2 \end{bmatrix} \sim \begin{bmatrix} \gamma+2 & 0 & \gamma+2 \\ 2(\gamma+2) & 0 & 2(\gamma+2) \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the eigenvector equation is $x + z = 0$. The general solution of this equation is

$$\begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad s \text{ and } t \text{ are free parameters.}$$

We therefore get the two basic eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For the eigenvalue -2 we have

$$-2I_3 - A = \begin{bmatrix} \gamma+2 & 0 & 2(\gamma+2) \\ -(\gamma+2) & -(\gamma+2) & -(\gamma+2) \\ -(\gamma+2) & 0 & -2(\gamma+2) \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding homogeneous linear system is therefore $x + 2z = 0$, $y = z$. Its general solution is

$$\begin{bmatrix} -2t \\ t \\ t \end{bmatrix}, \quad \text{hence we have one basic eigenvector:} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(b) The matrix is diagonalizable, since we found 3 basic eigenvectors. Matrices P and D as required for diagonalization are

$$P = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

10. (a) (1 bonus points) Let A be an $n \times n$ matrix. Give the definition of invertibility of A.

(b) (1 bonus points) Let A be an invertible $n \times n$ matrix. Use the Product Theorem for determinants to show that $\det(A) \neq 0$ and

$$\det(A^{-1}) = \frac{1}{\det(A)}\,.$$

Answer: (a) By definition, A is invertible if there exists a matrix B such that $AB = I_n = BA$, where I_n is the $n \times n$ identity matrix. In this case B is unique and denoted $B = A^{-1}$.

(b) Let A be invertible. Thus $AA^{-1} = I_n$. Since $\det(I_n) = 1$, the product theorem for determinants (Theorem 1 on page 82) implies $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Hence $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$.