1. (2 points) Find the complex number z satisfying

$$(2+3i)z = (1-2i)(3+2i).$$

A. $z = \frac{2}{13} - \frac{29}{13}i$
B. $z = \frac{2}{13} + \frac{29}{13}i$
C. $z = 2 - 29i$
D. $z = 2 + i$
E. $z = 2 - i$
F. $z = \frac{2}{13} - \frac{1}{13}i$

$$z = \frac{(1-2i)(3+2i)}{2+3i} = \frac{(3-4i^2)-6i+2i}{2+3i}$$
$$= \frac{7-4i}{2+3i} = \frac{(7-4i)(2-3i)}{(2+3i)(2-3i)} = \frac{1}{4+9}(14+12i^2-8i-21i)$$
$$= \frac{1}{13}(2-29i)$$

My answer: A

2. (2 points) Find all complex solutions z of the equation

$$z^2 - 6z + 34 = 0.$$

A. There are two solutions : $z = \frac{3}{2} + \frac{5}{2}i$ and $z = \frac{3}{2} - \frac{5}{2}i$
B. There are two solutions : $z = 3 + 5i$ and $z = 3 - 5i$
C. There are two solutions : $z = 1 + \frac{5}{2}i$ and $z = 1 - \frac{5}{2}i$
D. There are two solutions : $z = 2 + 5i$ and $z = 2 - 5i$
E. There is only solution : $z = 1 + \sqrt{2} + 5i$
F. There is only one solution : $z = 1 - \sqrt{2} + 5i$

$$0 = z^2 - 6z + 34 = (z-3)^2 - 9 + 34 = (z+3)^2 + 25$$
$$\Leftrightarrow (z-3)^2 = -25 \Leftrightarrow z = 3 \pm 5i$$

My answer: B

3. (2 points) Find the area of the triangle with vertices

$$A(2,1,-3),\quad B(1,2,-1),\quad \text{and}\quad C(1,1,0).$$

A. $\sqrt{91}$
B. $\sqrt{91}/2$
C. $\sqrt{91}/4$
D. $\sqrt{11}/8$
E. $\sqrt{11}/4$
F. $\sqrt{11}/2$

$\overrightarrow{AB} = [1\ 2\ -1]^T - [2,1,-3]^T = [-1,1,2]^T$

$\overrightarrow{AC} = [1\ 1\ 0]^T - [2,1,-3]^T = [-1,0,3]^T$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ -1 & 0 & 3 \end{vmatrix} = \left[\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} \right]^T$

$= [3,1,1]^T$

area of $\Delta = \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \frac{1}{2}\sqrt{9+1+1} = \frac{1}{2}\sqrt{11}$

My answer: F

4. (2 points) Which of the following is an equation for the plane containing the point $A(1,1,2)$ and the line with scalar equations $x = 3+2t$, $y = t$, $z = 1$?

A. $x + 3y - 2z = 1$
B. $x + 5y - 3z = 0$
C. $2x + y = 3$
D. $2x + 3y + z = 7$
E. $3x + z = 3$
F. $x - 2y + 4z = 7$

We take two points on the line, say $B(3,0,1)$ ($t=0$) and $C(5,1,1)$ (for $t=1$). Hence $A(1,1,2)$, $B(3,0,1)$ and $C(5,1,1)$ lie in the plane

So $\overrightarrow{AB} \times \overrightarrow{AC}$ is a normal of the plane:

$\overrightarrow{AB} \times \overrightarrow{AC} = ([3,0,1]^T - [1,1,2]^T) \times ([5,1,1]^T - [1,1,2]^T) = [2,-1,-1]^T \times [4,0,-1]^T$

$= \begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 4 & 0 & -1 \end{vmatrix} = \left[\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 4 & -1 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} \right]^T =$

$= [1,-2,4]^T$

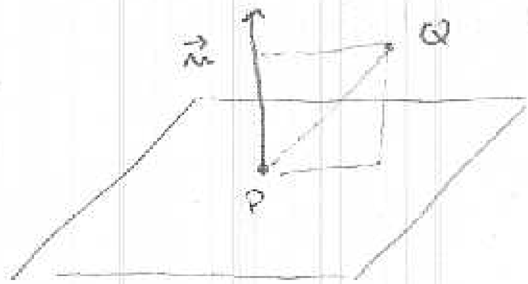
So the plane has the equation $x - 2y + 4z = k$ where k is a constant. Since A lies in the plane: $k = 1 - 2\cdot 1 + 4 \cdot 2 = 7$

Thus $x - 2y + 4z = 7$ is the equation of the plane

My answer: F

5. (2 points) What is the shortest distance from the point $Q(1,-1,1)$ to the plane with equation $3x+y-z=3$?

A. $2\sqrt{11}$
B. $2/\sqrt{11}$
C. $2\sqrt{3}$
D. $2/\sqrt{3}$
E. $2/11$
F. 22



The shortest distance is the length of the vector $\text{proj}_{\vec{n}}(\vec{PQ})$ where P is some point in the plane, say $P(1,0,0)$

Then $\vec{PQ} = [1,-1,1]^T - [1,0,0]^T = [0,-1,1]^T$, $\vec{n} = [3,1,-1]^T$, so

$$\text{proj}_{\vec{n}}(\vec{PQ}) = \frac{\vec{PQ}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{[0,-1,1]\cdot[3,1,-1]}{9+1+1}\vec{n} = -\frac{2}{11}[3,1,-1]^T$$

Hence $\|\text{proj}_{\vec{n}}(\vec{PQ})\| = \frac{2}{11}\|[3,1,-1]^T\| = \frac{2}{11}\sqrt{9+1+1} = 2\frac{\sqrt{11}}{11} = \frac{2}{\sqrt{11}}$

My answer: B

6. (2 points) Which of the following lines goes through the two points $P(1,-2,2)$ and $Q(2,0,-1)$?

A. $\mathbf{p} = [1\ -2\ 2]^T + t[2\ 0\ -1]^T$
B. $\mathbf{p} = [3\ -2\ 1]^T + t[1\ 2\ -3]^T$
C. $\mathbf{p} = [3\ -2\ 1]^T + t[2\ 0\ -1]^T$
D. $\mathbf{p} = [2\ 0\ -1]^T + t[1\ 2\ 1]^T$
E. $\mathbf{p} = [3\ 2\ -4]^T + t[1\ 2\ -3]^T$
F. $\mathbf{p} = [3\ 2\ -4]^T + t[1\ 2\ 1]^T$

A direction vector of the plane is $\vec{PQ}$
$= [2,0,-1]^T - [1,-2,2]^T = [1,2,-3]^T$,
so only B or E are possible.

Check if P lies on the line in (B):

$$[1,-2,2]^T \stackrel{?}{=} [3,-2,1]^T + t[1,2,-3]^T = [3+t,-2+2t,1-3t]^T$$

$$\Leftrightarrow \begin{matrix} 3+t=1 \\ -2+2t=-2 \\ 1-3t=2 \end{matrix} \Leftrightarrow \begin{matrix} t=-2 \\ t=0 \\ t=-\frac{1}{3} \end{matrix} \rightarrow \text{no solution}$$

Check if P lies on the line in (E):

$$[1,-2,2]^T = [3,2,-4]^T + t[1,2,-3]^T = [3+t,2+2t,-4-3t]^T$$

$$\Leftrightarrow \begin{matrix} 3+t=1 \\ 2+2t=-2 \\ -4-3t=2 \end{matrix} \Leftrightarrow t=-2 \quad \text{so (E) is correct.}$$

My answer: E

7. (2 points) Find the matrix A satisfying

$$\left(2A^T - 3\begin{bmatrix} 4 & 0 \\ 4 & 3 \\ 0 & -1 \end{bmatrix}\right)^T = \begin{bmatrix} -4 & 0 & 4 \\ 5 & 7 & 0 \end{bmatrix}.$$

The first row of A is

A. $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

B. $[3 \quad -1 \quad 2]$.

C. $[4 \quad 6 \quad 2]$.

D. $\begin{bmatrix} 20 \\ 9 \end{bmatrix}$.

E. There does not exist such a matrix.

F. $[4 \quad 0 \quad -8]$.

$$\begin{bmatrix} -4 & 0 & 4 \\ 5 & 7 & 0 \end{bmatrix} = 2(A^T)^T - 3\begin{bmatrix} 4 & 0 \\ 4 & 3 \\ 0 & -1 \end{bmatrix}^T = 2A - 3\begin{bmatrix} 4 & 4 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A = \frac{1}{2}\left(\begin{bmatrix} -4 & 0 & 4 \\ 5 & 7 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 12 & 0 \\ 0 & 9 & -3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 8 & 12 & 4 \\ 5 & 16 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 2 \\ 5/2 & 8 & -3/2 \end{bmatrix}$$

My answer: ________________

8. (2 points) In the matrix D below, replace α by the **last** digit of your student number. Consider the matrices

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -5 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 & 2 & -3 & 5 \\ 0 & 0 & 3 & 0 & -2 \\ 1 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}.$$

Among the matrices A, B, C, D, E, the matrix/matrices in reduced row-echelon form is/are

B only

A not because of position (22)

C __________ (12) or (15)

D __________ (3,1)

E __________ (21)

My answer: B

9. (5 points) Find the reduced row echelon form of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 4 & -5 & 3 & 4 & -\beta \\ -3 & 2 & 0 & -8 & 2 \end{bmatrix}$$

where β is the second last digit of your student number, and determine its rank.

Answer: After exchange of the first and second row we get

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 1 \\ 4 & -5 & 3 & 4 & -\beta \\ -3 & 2 & 0 & -8 & 2 \end{bmatrix} \overset{(1)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & -5 & 3 & -4 & -\beta \\ 0 & 2 & 0 & -2 & 2 \end{bmatrix}$$

$$\overset{(2)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 8 & -24 & 5-\beta \\ 0 & 0 & -2 & 6 & 0 \end{bmatrix} \overset{(3)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 5-\beta \end{bmatrix}$$

$$\text{for } \beta = 5: \quad \overset{(4)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{for } \beta \neq 5: \quad \overset{(5)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Explanations:

(1) Subtract multiples of rot 1 from the rows below.
(2) Subtract multiples of row 2 from the rows below.
(3) Exchange row 3 and row 4; divide the new row 3 by -2 and subtract a multiple from row 4.
(4) create a 0 in the position (23)
(5) Divide by $5-\beta$ and create a 0 in the position (25).
The rank is 3 for $\beta = 5$ and 4 for $\beta \neq 5$.

Marking: 4 points for correct reduction; 1 point for rank based on the rref the student obtained

Answer: The reduced row echelon form is

The matrix has rank:

10. (7 points) Determine the values of a for which the linear system

$$\begin{array}{rcrcrcl} 3x & - & 6y & + & 5z & = & 9 \\ & & ay & - & 3z & = & 0 \\ & & & & a(a-1)z & = & a \end{array}$$

has (i) infinitely many solutions, (ii) a unique solution, (iii) no solution.
In case (i) determine all solutions.

Answer: If $a = 0$ the system becomes

$$\begin{array}{rcrcrcl} 3x & - & 6y & + & 5z & = & 9 \\ & & & - & 3z & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

which is a system with infinitely many solutions, namely

$$x = 2t + 3, \quad y = t \text{ free parameter}, \quad z = 0.$$

We can therefore in the following assume that $a \neq 0$ and can then divide the last equation by a. The new system is

$$\begin{array}{rcrcrcl} 3x & - & 6y & + & 5z & = & 9 \\ & & ay & - & 3z & = & 0 \\ & & & & (a-1)z & = & 1 \end{array}$$

For $a = 1$ the last equation is $0z = 1$. Hence the system becomes inconsistent. For $a \neq 1$ and $a \neq 0$ the system is uniquely solvable. We get

$$z = \frac{1}{a-1}, \quad y = 3z = \frac{3}{a-1}, \quad x = \frac{1}{3}(9 + 6y - 5z = 9 + \frac{13}{a-1}) = 3 + \frac{13}{3(a-1)}.$$

(It was not required to find the unique solutions)

Marking: 2 points for each case, with explanation; 1 point for finding all solutions in case (i)

Answer: Case (i): **Case (ii):** **Case (iii)**

All solutions in case (i):