University of Ottawa

Department of Mathematics and Statistics

MAT 1341 B: Introduction to Linear Algebra

Instructor: Erhard Neher

Assignment 5; due November 15, 2007, 17:30 in the class room

Family Name: ______________________________

First Name: ______________________________

Student number: ___________________________

The last digit of your student number is $\alpha =$

The second last digit of your student number is $\beta =$

Please read these instructions carefully:

- Enter your name on this page and the next, but your student number only on this page. For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages of the assignment.
- The table below is for the TA. Do not write in the table.

Quest.	1	2	3	4	5	Total
maximal	4	8	4	4	6	26
score						

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Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
- For all questions you must show your work to obtain the points. Simply writing the correct answer will earn you 0.
- Please write legibly and argue logically: You must convince the TA that you know why your solution is correct.
- You have to submit this assignment at the beginning of the DGD on Thursday, November 15, 2007, at 17:30 in the classroom, at the latest. If you wish to submit it earlier, please do so at the secretariat of the Department of Mathematics, room 103A, 8:45–12:00 and 13:00–17:00, or at the beginning of my Thursday lecture.

1. (4 points) (a) (2 points) Can a 3×4 matrix have linearly independent columns? Linearly independent rows?

(b) (2 points) Can the null space of a 3×6 matrix have dimension 2? Give a reasoned answer.

Answer: A 3×4 matrix has 4 columns, each of them a vector of $\mathbb{R}^3$. Since the maximum number of linearly independent vectors of $\mathbb{R}^3$ is 3, the columns cannot be linearly independent. Another reason: $\dim col(A) = \dim row(A) = rank(A)$. The rank of A can be at most 3. Hence, $\dim col(A) \leq 3$.

Yes, the rows of A can be linearly independent. For example, this is so for

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

But note that the rows do not have to be linearly independent. Example: The rows of the zero matrix are linearly dependent.

(b) Suppose the null space has dimension 2. We know (Theorem 6 on page 211) that its dimension is $n - r$, where A is an $m \times n$ matrix and $r = \text{rank}(A)$. Thus, in our example, $n = 6$ and therefore $2 = n - r = 6 - r$ forces $r = 6 - 2 = 4$. But the rank of a 3×6 matrix is at most 3, contradiction. So the answer to the question is no: The null space of a 3×6 matrix cannot have dimension 2.

2. (8 points) Let α denote the **last digit** of your student number. Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 2 & 3 & 4 & 1 & 0 \\ -1 & -1 & -5 & 3 & -\alpha \\ 0 & 1 & -6 & 7 & 10 \end{bmatrix}.$$

(a) (2 points) Find the reduced row echelon form of A.
(b) (1 point) Find a basis for $row(A)$.
(c) (1 point) Find a basis for $col(A)$.
(d) (2 points) Find a basis for $null(A)$.
(e) (2 points) Give the dimensions of these three subspaces and verify the rank theorem (page 205) and the corollary about null space and image (page 209) for this matrix.

Answer: To answer (a) through (d) we need to reduce the matrix to RREF.

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 2 & 3 & 4 & 1 & 0 \\ -1 & -1 & -5 & 3 & -\alpha \\ 0 & 1 & -6 & 7 & 10 \end{bmatrix} \underset{\substack{-2R1+R2 \\ R1+R2}}{\sim} \begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 0 & -1 & 6 & -7 & -10 \\ 0 & 1 & -6 & 7 & 5-\alpha \\ 0 & 1 & -6 & 7 & 10 \end{bmatrix}$$

$$\underset{\substack{2R2+R1 \\ -R2 \\ R2+R3 \\ R2+R4}}{\sim} \begin{bmatrix} 1 & 0 & 11 & -10 & -15 \\ 0 & 1 & -6 & 7 & 10 \\ 0 & 0 & 0 & 0 & -5-\alpha \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, divide row 3 by $-5-\alpha$ (which is nonzero) and clear out the last column to get

$$\sim \begin{bmatrix} 1 & 0 & 11 & -10 & 0 \\ 0 & 1 & -6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

This matrix B is in RREF.

(b) The row space of A is equal to the row space of B. Since the nonzero rows of B are linearly independent and span row(A) we deduce that a basis for row(A) is

$$\{[1 \quad 0 \quad 11 \quad -10 \quad 0]^T, [0 \quad 1 \quad -6 \quad 7 \quad 0]^T, [0 \quad 0 \quad 0 \quad 0 \quad 1]^T\}$$

(c) The column space of A (or image of A) has basis given by the columns of A which correspond to columns of B containing leading ones. These are the first, second and fifth columns of B. So a basis is

$$\{[1 \quad 2 \quad -1 \quad 0]^T, [2 \quad 3 \quad -1 \quad 1]^T, [5 \quad 0 \quad -\alpha, 10]^T\}$$

(d) The nullspace of A is the set of all solutions to $AX = 0$, so we augment the above reduction by the zero vector and read off the solution as in Chapter 1:

$$x_1 + 11x_3 - 10x_4 = 0; \; x_2 - 6x_3 + 7x_4 = 0; \; x_5 = 0$$

Setting our nonleading variables equal to parameters s, t we deduce that the solutions all have the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -11 \\ 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 10 \\ -7 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

for some choice of r, s. Thus these two basic solutions span $null(A)$ and they are linearly independent so they form a basis for $null(A)$.

(e) We have $\dim(row(A)) = 3 = \dim(col(A))$, as we expected from our rank theorem. Moreover, $dim(null(A)) = 2$, so $\dim(Im(A)) + \dim(null(A)) = n = 5$, as predicted by the Corollary.

3. (4 points) Let $U = \{A \in \mathbb{M}_{4,4} : A^T = -A\}$.

(a) (1 point) Give an example of a non-zero matrix in U.

(b) (3 points) Show that U is a subspace of $\mathbb{M}_{4,4}$ by verifying the 3 conditions of the subspace test (Theorem 3 on page 297). Hint: You do not need to write out 4×4 matrices to answer this question. Instead, use matrix algebra (Chapter 1.1).

Answer: (a) Let $A = [a_{ij}] \in \mathbb{M}_{4,4}$. Then $A^T = -A$ if and only if $a_{ij} = -a_{ji}$ holds for all i, j. Note that in particular $a_{ii} = -a_{ii}$ shows $a_{ii} = 0$. It is now easy to write down a matrix in U. There are literally millions of non-zero matrixes in U, for example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (S1) The zero matrix $\mathbf{0}$ is in U since $\mathbf{0}^T = \mathbf{0} = -\mathbf{0}$.

(S2) Let $A \in U$ and $B \in U$. Thus $A^t = -A$ and $B^T = -B$. Then, by Theorem 4 of section 1.1, page 8 (didn't I tell you that we'll need this!), we have $(A+B)^T = A^T + B^T = (-A) + (-B) = -(A+B)$ proving that $A + B \in U$.

(S3) Let $A \in U$, i.e., $A^T = -A$ and let $a \in \mathbb{R}$. Then, by the Theorem just quoted, $(aA)^T = a(A^T) = a(-A) = -(aA)$, so $aA \in U$.

4. (4 points) Let U be the set of 2×2 non-invertible matrices:

$$U = \{A \in \mathbb{M}_{2,2} : \det(A) = 0\}$$

This is a subset of the vector space $\mathbb{M}_{2,2}$.
(a) (2 points) Give an example of two matrices which are in the set U but whose sum is not in U.
(b) (2 points) Is U a subspace of $\mathbb{M}_{2,2}$? Justify your answer.

Answer: (a) For example, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The first two matrices are in the set U because they have determinant zero. Their sum is the identity matrix, which has determinant equal to 1, and so is not in U.

(b) No, U is not a subspace, because it fails the subspace test. That is, it is NOT TRUE that for every pair of elements X, Y in U, their sum $X + Y$ is in U.

5. (6 points) Let β be the **second last** digit of your student number. Let $\mathbb{P}_3$ denote the vector space consisting of all polynomials of degree at most 3. Let U be the set of polynomials in $\mathbb{P}_3$ which have β as a root, that is,

$$U = \{p(x) \in \mathbb{P}_3 : p(\beta) = 0\}.$$

Let V be the set of all polynomials in $\mathbb{P}_3$ such that when evaluated at 0 the answer is $\beta + 1$:

$$V = \{p(x) \in \mathbb{P}_3 : p(0) = \beta + 1\}$$

(a) Is U a subspace of $\mathbb{P}_3$? Justify your answer.
(b) Is V a subspace of $\mathbb{P}_3$? Justify your answer.

Answer: (a) We check the subspace test.

(1) Is the zero vector in U? For the vector space of polynomials, the zero vector is the polynomial with all zero coefficients; it is zero on every point so in particular it satisfies $p(\beta) = 0$. Hence it lies in U.

(2) If p, q are in U, is $p + q$ in U? The way to decide if a given polynomial is in U is to evaluate it at β; the polynomial is in U if and only if the answer is zero. So we are given that $p(\beta) = 0$ and $q(\beta) = 0$. By definition, $(p + q)(\beta) = p(\beta) + q(\beta) = 0 + 0 = 0$, so $p + q$ is indeed in U.

(3) If p is in U and k is a scalar, is kp in U? Again, if p is in U then $p(\beta) = 0$; but then $(kp)(\beta) = kp(\beta) = k0 = 0$ so kp is in U.

Thus U satisfies the subspace test; it is a subspace of $\mathbb{P}_3$.

(b) Since $\beta + 1 \neq 0$, we see immediately that the zero polynomial does not lie in V. Hence V fails the subspace test; it is not a subspace.