University of Ottawa

Department of Mathematics and Statistics

MAT 1341 B: Introduction to Linear Algebra

Instructor: Erhard Neher

Assignment 4; due November 8, 2007, 17:30 in the class room

Family Name: ____________________________

First Name: ____________________________

Student number: ____________________________

The last digit of your student number is $\alpha =$

The second last digit of your student number is $\beta =$

The third last digit of your student number is $\gamma =$

Please read these instructions carefully:

- Enter your name on this page and the next, but your student number only on this page. For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages of the assignment.
- Question 3 is a more challenging question. This question does not count towards the total marks of the homework. You will get 5 extra points for a complete and correct solution.
- The table below is for the TA. Do not write in the table.

Quest.	1	2	3	Total
maximal	8	8	5 extra points	16
score				

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Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
- For all questions you must show your work to obtain the points. Simply writing the correct answer will earn you 0.
- Please write legibly and argue logically: You must convince the TA that you know why your solution is correct.
- You have to submit this assignment at the beginning of the DGD on Thursday, November 8, 2007, at 17:30 in the classroom, at the latest. If you wish to submit it earlier, please do so at the secretariat of the Department of Mathematics, room 103A, 8:45–12:00 and 13:00–17:00, or at the beginning of my Thursday lecture.

1. (8 points) For each of the following sets of vectors, answer two questions: (i) Do the vectors span $\mathbb{R}^3$? (ii) Is the set of vectors linearly independent? In each case, **justify your answer** and, if the vectors are linearly dependent, express one vector as a linear combination of the others. (Please note that a YES or NO answer to any question without a correct justification will not earn you any marks.)

(a) $\{[1,2,3]^T, [1,1,2]^T\}$
(b) $\{[1,2,3]^T, [1,1,2]^T, [2,1,0]^T\}$
(c) $\{[1,2,3]^T, [1,1,2]^T, [2,1,0]^T, [1,0,1]^T\}$

Answer: (a) We have only two vectors and by the Fundamental Theorem (Section 4.3, page 193), two vectors is not enough to span all of $\mathbb{R}^3$. These vectors are nonzero and are not parallel, however, so they are linearly independent. So: (i) NO, (ii) YES.

(b) Since this is a set of 3 vectors in $\mathbb{R}^3$, we can apply Theorem 2 of Section 4.2 (page 192). That is, form a matrix A with these vectors as its column vectors and decide whether or not it is invertible. We have

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}.$$

We have many choices for checking the invertibilitiy of A, including row reducing A (and seeing if A has full rank, equivalently, that there are no zero rows in the RREF of A) or taking the determinant of A (and seeing if it is nonzero). Let's go for the determinant:

$$\det \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & -1 & -6 \end{bmatrix} = 3(6-3) = 9 \neq 0$$

Hence A is invertible, so by the theorem, its columns span $\mathbb{R}^3$ and are linearly independent. (Hence this set is a basis for $\mathbb{R}^3$.) So: (i) YES, (ii) YES.

(c) Since the three vectors given in part (b) are included in the generating set for part (c), we can apply Theorem 1 from Section 4.1 (page 185). That is, since X_1, X_2, X_3 are in $span\{X_1, X_2, X_3, X_4\}$, the whole of $span\{X_1, X_2, X_3\}$ must be contained in $span\{X_1, X_2, X_3, X_4\}$. But in part (b) we deduced that $span\{X_1, X_2, X_3\} = \mathbb{R}^3$ (and you can't get any bigger than that) so we must have $span\{X_1, X_2, X_3, X_4\} = \mathbb{R}^3$.

On the other hand, by the Fundamental Theorem (page 195), we know there are too many vectors (4 vectors in $\mathbb{R}^3$), and so they must be linearly dependent. To write one of these vectors as a linear combination of the others, we will find all solutions to $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$ and work from there. So we reduce the matrix $[A|0]$:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & | & 0 \\ 2 & 1 & 1 & 0 & | & 0 \\ 3 & 2 & 0 & 1 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & | & 0 \\ 0 & -1 & -3 & -2 & | & 0 \\ 0 & -1 & -6 & -2 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 1 & | & 0 \\ 0 & 1 & 3 & 2 & | & 0 \\ 0 & 0 & -3 & 0 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & 2 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \end{bmatrix}$$

(where we have done several row reduction steps for the final solution). Thus our general solution is

$$c_1 = t, \quad c_2 = -2t, \quad c_3 = 0, \quad , c_4 = t$$

for any t. This means that for any t:

$$tX_1 - 2tX_2 + 0X_3 + tX_4 = 0.$$

In particular, setting $t = 1$ we can solve for (for example) X_4:

$$X_4 = -X_1 + 2X_2 = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

as required.

2. Let α, β, γ be the last, the second last and third last digit of your student number, and let

$$U = \{ [\, x_1 \quad x_2 \quad x_3 \quad x_4 \,]^T : x_1 + \alpha x_2 + \beta x_3 + \gamma x_4 = 0 \text{ and } x_2 + \beta x_3 + \gamma x_4 = 0 \}.$$

(a) (3 points) Show that U is a subspace of $\mathbb{R}^4$.

(b) (4 points) Find a basis of U.

(c) (1 points) What is the dimension of U?

Answer: (a) Note that $x_2 = -\beta x_3 - \gamma x_4$. We substitute this in the equation for x_1 and get

$$\begin{aligned} x_1 &= -\alpha x_2 - \beta x_3 - \gamma x_4 = -\alpha(-\beta x_3 - \gamma x_4) - \beta x_3 - \gamma x_4 \\ &= \alpha\beta x_3 + \alpha\gamma x_4 - \beta x_3 - \gamma x_4 = \beta(\alpha - 1)x_3 + \gamma(\alpha - 1)x_4 \end{aligned}$$

Hence

$$\begin{aligned} U &= \{ [\, x_1 \quad x_2 \quad x_3 \quad x_4 \,]^T : x_1 = \beta(\alpha-1)x_3 + \gamma(\alpha-1)x_4 \text{ and } x_2 = -\beta x_3 - \gamma x_4 \} \\ &= \{ [\, \beta(\alpha-1)x_3 + \gamma(\alpha-1)x_4, \quad -\beta x_3 - \gamma x_4, \quad x_3, \quad x_4 \,]^T : x_3, x_4 \in \mathbb{R} \} \\ &= \{ x_3 [\, \beta(\alpha-1), \quad -\beta, \quad 1, \quad 0 \,] + x_4 [\, \gamma(\alpha-1), \quad -\gamma, \quad 0, \quad 1 \,] : x_3, x_4 \in \mathbb{R} \} \\ &= \mathit{span} \{ [\, \beta(\alpha-1), \quad -\beta, \quad 1, \quad 0 \,]^T, [\, \gamma(\alpha-1), \quad -\gamma, \quad 0, \quad 1 \,]^T \}. \end{aligned}$$

Since the span of a set of vectors is always a subspace (Th.1, page 185), U is a subspace.

2nd solution of (a): Check the three conditions (S1), (S2) and (S3) defining a subspace:

(S1) The zero vector $0 \in U$ since the components of 0 are all 0 and hence satisfy the conditions defining U, e.g. $0 + \alpha 0 + \beta 0 + \gamma 0 = 0$.

(S2) Suppose $X = [\, x_1 \quad x_2 \quad x_3 \quad x_4 \,]^T \in U$ and $Y = [\, y_1 \quad y_2 \quad y_3 \quad y_4 \,]^T \in U$. Then $X + Y = [\, x_1 + y_1 \quad x_2 + y_2 \quad x_3 + y_3 \quad x_4 + y_4 \,]^T$ lies in U if the components of $X + Y$ satisfy the conditions defining U. From $X, Y \in U$ we have

$$\begin{aligned} x_1 + \alpha x_2 + \beta x_3 + \gamma x_4 &= 0 \\ y_1 + \alpha y_2 + \beta y_3 + \gamma y_4 &= 0 \quad \text{hence} \\ (x_1 + y_1) + \alpha(x_2 + y_2) + \beta(x_3 + y_3) + \gamma(x_4 + y_4) &= 0. \end{aligned}$$

Similarly for the second condition:

$$\begin{aligned} x_2 + \beta x_3 + \gamma x_4 &= 0 \\ y_2 + \beta y_3 + \gamma y_4 &= 0 \quad \text{hence} \\ (x_2 + y_2) + \beta(x_3 + y_3) + \gamma(x_4 + y_4) &= 0. \end{aligned}$$

(S3) Let again $X = [\, x_1 \quad x_2 \quad x_3 \quad x_4 \,]^T \in U$ and let $s \in \mathbb{R}$. Then $sX = [\, sx_1 \quad sx_2 \quad sx_3 \quad sx_4 \,]^T$, and this vector lies in U if the components satisfy the conditions defining U:

$$sx_1 + \alpha s x_2 + \beta s x_3 + \gamma s x_4 = s(x_1 + \alpha x_2 + \beta x_3 + \gamma x_4) = s0 = 0$$
$$sx_2 + \beta s x_3 + \gamma s x_4 = s(x_2 + \beta x_3 + \gamma x_4) = s0 = 0,$$

proving $sX \in U$.

3rd solution of (a): Let

$$A = \begin{bmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & \alpha & \gamma \end{bmatrix}.$$

Then $U = \mathit{null}\, A$, and is therefore a subspace of $\mathbb{R}^4$ (Example 6 on page 184).

(b) By the first solution of (a), we know that U is spanned by the two vectors

$$[\,\beta(\alpha-1), \quad -\beta, \quad 1, \quad 0\,]^T, \quad [\,\gamma(\alpha-1), \quad -\gamma, \quad 0, \quad 1\,]^T.$$

We claim that these two vectors are also linearly independent. Indeed, suppose $s, t \in \mathbb{R}$ and

$$s\,[\,\beta(\alpha-1), \quad -\beta, \quad 1, \quad 0\,]^T + t\,[\,\gamma(\alpha-1), \quad -\gamma, \quad 0, \quad 1\,]^T = 0$$

Since (see above)

$$\begin{aligned} s\,[\,\beta(\alpha-1), \quad -\beta, \quad 1, \quad 0\,]^T &+ t\,[\,\gamma(\alpha-1), \quad -\gamma, \quad 0, \quad 1\,]^T \\ = [\,\beta(\alpha-1)s + \gamma(\alpha-1)t, &\quad -\beta s - \gamma t, \quad s, \quad t\,]^T \end{aligned}$$

we get $s = 0 = t$ by looking at the third and fourth component. Thus,

$$B = \{\,[\,\beta(\alpha-1), \quad -\beta, \quad 1, \quad 0\,]^T, [\,\gamma(\alpha-1), \quad -\gamma, \quad 0, \quad 1\,]^T\}$$

is a basis of U.

(c) $\dim U = 2$, since B has 2 elements.

3. (**5 bonus points**) If $X, Y \in \mathbb{R}^4$ are two linearly independent vectors, show that there exist $s, t \in \{1, 2, 3, 4\}$ such that $\{X, Y, E_s, E_t\}$ is a basis of $\mathbb{R}^4$ (recall that $\{E_1, E_2, E_3, E_4\}$ is the standard basis of $\mathbb{R}^4$).

Answer: $\{E_1, E_2, E_3, E_4\}$ is not contained in $span\,\{X, Y\}$. Indeed, assume that $\{E_1, E_2, E_3, E_4\} \subset span\,\{X, Y\}$. Then $\mathbb{R}^4 = span\,\{E_1, E_2, E_3, E_4\} \subset span\,\{X, Y\} \subset \mathbb{R}^4$ by Theorem 1(2) on page 185. It would follow that $span\,\{X, Y\} = \mathbb{R}^4$, thus $\{X, Y\}$ is a basis of $\mathbb{R}^4$, contradicting $\dim \mathbb{R}^4 =$. So some E_s is not in $span\,\{X, Y\}$. By the Independence Lemma, $\{X, Y, E_s\}$ is linearly independent.

In the same way, $\{E_1, E_2, E_3, E_4\}$ is not contained in $span\,\{X, Y, E_s\}$ since $\dim span\,\{X, Y, E_s\} = 3$ and is therefore a proper subspace of $\mathbb{R}_4$. So some E_t is not in $span\,\{X, Y, E_s\}$. Again by the Independence Lemma, $\{X, Y, E_s, E_t\}$ is linearly independent. Because $\dim \mathbb{R}^4 = 4$, any set of 4 linearly independent vectors is a basis (Theorem 4, page 198). Therefore $\{X, Y, E_s, E_t\}$ is a basis of $\mathbb{R}^4$.