1. (5 points) In the matrix below replace β with the **second last** digit of your student number and calculate its determinant.

$$\begin{bmatrix} 3 & 5 & -2 & \beta \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Answer:

$$\begin{vmatrix} 3 & 5 & -2 & \beta \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \overset{(a)}{=} \begin{vmatrix} 0 & -1 & 1 & \beta-3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \overset{(b)}{=} - \begin{vmatrix} -1 & 1 & \beta-3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \overset{(c)}{=} - \begin{vmatrix} 0 & 9 & \beta-3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \overset{(d)}{=} - \begin{vmatrix} 9 & \beta-3 \\ 3 & 3 \end{vmatrix}$$

$$\overset{(e)}{=} -(27 - 3\beta + 9) = 3\beta - 36.$$

Explanations: (a) add suitable multiples of the second row to the other rows.
(b) cofactor expansion along the first column
(c) add the last row to the first row
(d) cofactor expansion along the first column
(e) calculate the 2×2 determinant

2. (10 points) In the matrix below replace α with the **last** digit of your student number.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3\alpha & -5\alpha-6 & -6\alpha-6 \\ -3\alpha-3 & 5\alpha+5 & 6\alpha+5 \end{bmatrix}.$$

(a) (3 points) Find the characteristic polynomial of A.

(b) (1 point) Find all eigenvalues of A.

(c) (4 points) For each eigenvalue find a set of basic eigenvectors.

(d) (2 points) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Justify your answer.

Answer: (a) We have

$$\lambda I - A = \begin{bmatrix} \lambda+1 & 0 & 0 \\ -3\alpha & \lambda+5\alpha+6 & 6\alpha+6 \\ 3\alpha+3 & -5\alpha-5 & \lambda-6\alpha-5 \end{bmatrix}.$$

Thus, by cofactor expansion on row 1, we get

$$\begin{aligned} \det(\lambda I - A) &= (\lambda+1) \det \begin{bmatrix} \lambda+5\alpha+6 & 6\alpha+6 \\ -5\alpha-5 & \lambda-6\alpha-5 \end{bmatrix} \\ &= (\lambda+1)((\lambda+5\alpha+6)(\lambda-6\alpha-5) - (6\alpha+6)(-5\alpha-5)) \\ &= (\lambda+1)(\lambda^2 + (-\alpha+1)\lambda - \alpha) \\ &= (\lambda+1)(\lambda+1)(\lambda-\alpha) \end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial: so -1 and α. (The eigenvalue -1 is a repeated root.)

(c) For $\lambda = \alpha$, we row reduce $[\alpha I - A|0]$:

$$\begin{bmatrix} \alpha+1 & 0 & 0 & | & 0 \\ -3\alpha & 6\alpha+6 & 6\alpha+6 & | & 0 \\ 3\alpha+3 & -5\alpha-5 & -5\alpha-5 & | & 0 \end{bmatrix} \underset{\sim}{1/(1+\alpha)R1} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -3\alpha & 6\alpha+6 & 6\alpha+6 & | & 0 \\ 3\alpha+3 & -5\alpha-5 & -5\alpha-5 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

(where we have done several row operations to get to this last step!) Thus the general solution is

$$x = 0, y + z = 0, z = t$$

which you can write as $[x\ y\ z]^T = t[0\ -1\ 1]^T$, so a basic solution is

$$X = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

This is an eigenvector for the eigenvalue α.

For $\lambda = -1$, we row reduce $[-I - A|0]$:

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ -3\alpha & 5\alpha+5 & 6\alpha+6 & | & 0 \\ 3\alpha+3 & -5\alpha-5 & -6\alpha-6 & | & 0 \end{bmatrix} \overset{\sim}{(1/(\alpha+1))R3} \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ -3\alpha & 5\alpha+5 & 6\alpha+6 & | & 0 \\ 3 & -5 & -6 & | & 0 \end{bmatrix}$$

$$\begin{matrix} R1 <-> R3 \\ 3\alpha R1 + R2 \\ \sim \end{matrix} \begin{bmatrix} 3 & -5 & -6 & | & 0 \\ 0 & 5 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \underset{\sim}{R2 + R1} \begin{bmatrix} 3 & 0 & 0 & | & 0 \\ 0 & 5 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 6/5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus the system is $x = 0$ and $y + 6/5z = 0$ so the solution is $[x\ y\ z]^T = t[0\ -6/5\ 1]^T$; thus the only basic solution is

$$X = \begin{bmatrix} 0 \\ -6/5 \\ 1 \end{bmatrix}$$

(d) Since we have found only 2 basic solutions for the 3×3-matrix, it is not diagonalizable.

3. (5 extra points) Let $x_1, x_2, \ldots, x_n$ be real numbers, $n \geq 2$. Show that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq j < i \leq n} (x_i - x_j)$$

where $\prod_{1 \leq j < i \leq n} (x_i - x_j)$ means the product of all factors $(x_i - x_j)$ for all pairs (i, j) satisfying $j < i$ and i and j between 1 and n. Hint: Replace the i^{th}-column C_i of the matrix in question by $C_i - x_1 C_{i-1}$, and expand along the first row. Then reduce to the analogous determinant for $n - 1$ numbers.

Answer: We follow the suggestion: Let A denote the above matrix. Add $-x_1$ times column C_{n-1} to column C_n, then $-x_1$ times column C_{n-2} to column C_{n-1}, and so forth, until we finally add $-x_1$ times column C_1 to column C_2. The resulting matrix is

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & x_2^{n-1} - x_1 x_2^{n-2} \\ 1 & x_3 - x_1 & x_3^2 - x_1 x_3 & \cdots & x_3^{n-1} - x_1 x_3^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n - x_1 & x_n^2 - x_1 x_n & \cdots & x_n^{n-1} - x_1 x_n^{n-2} \end{bmatrix}$$

By Theorem 2 on page 77, $\det(B) = \det(A)$. Now do the cofactor expansion along the first row; only the first term is nonzero, so we deduce

$$\det(A) = \det(B) = 1 \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & x_2^{n-1} - x_1 x_2^{n-2} \\ x_3 - x_1 & x_3^2 - x_1 x_3 & \cdots & x_3^{n-1} - x_1 x_3^{n-2} \\ \vdots & \vdots & & \vdots \\ x_n - x_1 & x_n^2 - x_1 x_n & \cdots & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix}.$$

Now we can rewrite this as

$$\begin{vmatrix} x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & (x_2 - x_1)x_2^{n-2} \\ x_3 - x_1 & (x_3 - x_1)x_3 & \cdots & (x_3 - x_1)x_3^{n-2} \\ \vdots & \vdots & & \vdots \\ x_n - x_1 & (x_n - x_1)x_n & \cdots & (x_n - x_1)x_n^{n-2} \end{vmatrix}$$

and so we can factor out $(x_2 - x_1)$ from the first row, $(x_3 - x_2)$ from the second row, $\cdots$, $(x_n - x_1)$ from the last row. By Theorem 2 page 77, we must multiply the determinant of the resulting matrix by these factors to preserve equality. That is,

$$\det(A) = (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \begin{vmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{vmatrix}$$

But now the determinant on the right is of the same type as the one we started with, only it's of size $(n-1) \times (n-1)$. Thus if we repeated the above argument $n - 2$ more times, we'd end up with

$$\det(A) = \prod_{j=2}^{n} (x_j - x_1) \cdot \prod_{j=3}^{n} (x_j - x_2) \cdots \prod_{j=n-1}^{n} (x_j - x_{n-1}) \det([1]) = \prod_{1 \leq j < i \leq n} (x_i - x_j)$$

as required.