1. (a) (2 points) In the matrices below replace α by the **last** digit of your student number and calculate all possible products between the two matrices

$$A = \begin{bmatrix} \alpha & 3 \\ 5 & 7 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 2 & 5 \end{bmatrix}$$

Answer: Since the format of A is 3×2 and the format of B is 2×3 we can form the products AB to get a 3×3-matrix and BA to get a 2×2-matrix:

$$AB = \begin{bmatrix} \alpha & 3 \\ 5 & 7 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -2 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} \alpha - 9 & 3\alpha + 6 & -2\alpha + 15 \\ -16 & 29 & 25 \\ -12 & 8 & 20 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} \alpha & 3 \\ 5 & 7 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \alpha + 15 & 16 \\ -3\alpha + 10 & 25 \end{bmatrix}$$

Marking: 1 point for each correct product.

(b) (2 points) Let A and B be arbitrary matrices. Which combination of true/false is correct for the following statements:

- If $AB = BA$ then A and B are both square and of the same size.
- If A has a row of zeros then also AB has a row of zeros.
- If $A^2 = A$ then either $A = 0$ or $A = I$.

(A) true, false, true
(B) false, true, false
(C) true, true, false
(D) true, false, false
(E) false, false, true
(F) false, false, false

Answer: Suppose A is an $m \times n$-matrix and B is an $p \times q$-matrix. Since AB exists, we must have $n = p$ and then AB is an $m \times q$-matrix. Since BA exists we must have $q = m$ and then BA is a $p \times m$-matrix. Now $AB = BA$ shows that $m = p$ and $q = n$, so $m = n = p = q$, and the first claim is correct.

The i^{th}-row if AB is obtained by multiplying the i^{th}-row of A with every column of B. Hence, if the i^{th}-row of A is zero, so is the i^{th}-row of AB. Thus, also the second statement is true.

Not true. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which has $A^2 = A$ but $0 \neq A \neq I$. Thus, the correct answer is (C).

Marking: 1 point if at least one answer corect, i.e., (B) or (D)

2. (5 points) In the matrix below replace β by the **second last** digit of your student number and find the inverse of the matrix

$$A = \begin{bmatrix} \beta & 1 & 2+3\beta \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

Please check your answer (1 point).

Answer: We apply the matrix inversion algorithm:

$$[A \quad I] = \begin{bmatrix} \beta & 1 & 2+3\beta & | & 1 & 0 & 0 \\ 1 & 0 & 3 & | & 0 & 1 & 0 \\ 4 & -3 & 8 & | & 0 & 0 & 1 \end{bmatrix} \overset{(a)}{\sim} \begin{bmatrix} 1 & 0 & 3 & | & 0 & 1 & 0 \\ \beta & 1 & 2+3\beta & | & 1 & 0 & 0 \\ 4 & -3 & 8 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{(b)}{\sim} \begin{bmatrix} 1 & 0 & 3 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & 1 & -\beta & 0 \\ 0 & -3 & -4 & | & 0 & -4 & 1 \end{bmatrix} \overset{(c)}{\sim} \begin{bmatrix} 1 & 0 & 3 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & 1 & -\beta & 0 \\ 0 & 0 & 2 & | & 3 & -4-3\beta & 1 \end{bmatrix}$$

$$\overset{(d)}{\sim} \begin{bmatrix} 1 & 0 & 3 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & 1 & -\beta & 0 \\ 0 & 0 & 1 & | & 3/2 & -2-3\beta/2 & 1/2 \end{bmatrix} \overset{(e)}{\sim} \begin{bmatrix} 1 & 0 & 0 & | & -9/2 & 7+9\beta/2 & -3/2 \\ 0 & 1 & 0 & | & -2 & 4+2\beta & -1 \\ 0 & 0 & 1 & | & 3/2 & -2-3\beta/2 & 1/2 \end{bmatrix}$$

Explanations: (a) exchange row 1 and row 2 since β could be zero, hence we cannot divide by β.

(b) Subtract multiples of the first row from the two rows below

(c) Add 3 times row 2 to row 3

(d) divide row 3 by 2

(e) add multiples of row 3 to the rows above.

Hence

$$A^{-1} = \begin{bmatrix} -9/2 & 7+9\beta/2 & -3/2 \\ -2 & 4+2\beta & -1 \\ 3/2 & -2-3\beta/2 & 1/2 \end{bmatrix}$$

Marking: 5 points for the correct inverse matrix; 1 point for checking the answer.

3. (6 points) Find basic solutions for the homogeneous linear system whose coefficient matrix is given below and express the general solution as a linear combination of these basic solutions.

$$\begin{bmatrix} 1 & -1 & 2 & \alpha \\ 2 & 2 & 0 & -1 \\ 3 & 1 & 2 & \alpha-1 \end{bmatrix}$$

Here α is the last digit of your student number.

Answer: We find the reduced row echelon form of the coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 2 & \alpha \\ 2 & 2 & 0 & -1 \\ 3 & 1 & 2 & \alpha-1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & \alpha \\ 0 & 4 & -4 & -1-2\alpha \\ 0 & 4 & -4 & -1-2\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & \alpha \\ 0 & 4 & -4 & -1-2\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 2 & \alpha \\ 0 & 1 & -1 & -(1+2\alpha)/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & (2\alpha-1)/4 \\ 0 & 1 & -1 & -(1+2\alpha)/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The new homogeneous system therefore is

$$\begin{aligned} x_1 + x_3 + (\alpha/2 - 1/4)x_4 &= 0 \\ x_2 - x_3 - (1/4 + \alpha/2)x_4 &= 0 \end{aligned}$$

Thus x_1 and x_2 are the leading variables and x_3, x_4 are the free variables. The general solution is as follows where $s, t \in \mathbb{R}$ are free parameters

$$\begin{bmatrix} -s-(\alpha/2-1/4)t \\ s+(1/4+\alpha/2))t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1/4-\alpha/2 \\ 1/4+\alpha/2 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

The equation above expresses the general solution as a linear combination of the two basic solutions

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1/4-\alpha/2 \\ 1/4+\alpha/2 \\ 0 \\ 1 \end{bmatrix}. \tag{2}$$

Marking: 3 points for the correct row reduction; it is not necessary to get a reduced ref. 1 point for the general solution and 1 point for expressing the general solution as a linear combination of basic solutions. 1 point for writing down the basic solutions.

The general solution expressed as a linear combination: see (1)

My basic solutions are: see (2)