1. (a) (2 points) Let α be the last digit of your student number. Write

$$\frac{\alpha + 5i}{7 - 3i}$$

in the form $a + ib$ where $a, b \in \mathbb{R}$.

$$\frac{\alpha + 5i}{7-3i} = \frac{(\alpha+5i)(7+3i)}{(7-3i)(7+3i)} = \frac{1}{49+9}\left[7\alpha - 15 + i(35+3\alpha)\right]$$

for example, for $\alpha = 1$ we get $-8 + 38i$

My answer: $7\alpha - 15 + i(35+3\alpha)$

(b) (2 points) Find all roots of $z^3 + 4z^2 + 6z = 0$

$z = 0$ is a root, we can therefore factor

$z^3 + 4z^2 + 6z = z(z^2 + 4z + 6)$. The roots of $z^2 + 4z + 6$

are $\dfrac{-4 \pm \sqrt{16 - 4\cdot 6}}{2} = -2 \pm \dfrac{\sqrt{-8}}{2} = -2 \pm \dfrac{2\sqrt{2}}{2} i =$

$= -2 \pm \sqrt{2} i$

My answer: $0, \; -2 \pm \sqrt{2}\, i$

2. (a) (3 points) Find the intersection point of the lines

$$\begin{aligned} x &= 1-3t \quad & x &= -1+s \\ y &= 2+5t \quad & y &= 3-4s \\ z &= 1+t \quad & z &= 1-s \end{aligned}$$

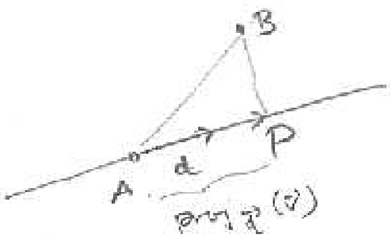
The intersection point is given by the conditions

$$\begin{aligned} x&=1-3t=-1+s \\ y&=2+5t=3-4s \\ z&=1+t=1-s \end{aligned} \Rightarrow \begin{aligned} s+3t&=2 \\ 4s+5t&=1 \\ s+t&=0 \end{aligned} \Rightarrow \begin{aligned} s-3s&=2 \\ 4s-5s&=1 \\ t&=-s \end{aligned} \Rightarrow \begin{aligned} s&=-1 \\ t&=-s \end{aligned}$$

$\Rightarrow$ $(s,t)=(-1,1)$ are the solutions of the linear system. Substituting $t=1$ into the equations for the line yields the point of intersection: $P(-2,7,2)$

My answer: $P(-2,7,2)$

(b) (4 points) Let L be the line through the point $A(3,1,1)$ with direction vector $\vec{d}=[-1 \quad 1 \quad 0]^T$. Find the point on the line L that is closest to the point $B(1,\beta,7)$ where β is the **second last digit** of your student number, and find the shortest distance from B to L.



$$\vec{v}=B-A=(1,\beta,7)-(3,1,1)=(-2,\beta-1,6)$$

$$\text{proj}_{\vec{d}}(\vec{v})=\frac{\vec{v}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d}=\frac{(-2,\beta-1,6)\cdot(-1,1,0)}{(-1,1,0)\cdot(-1,1,0)}\vec{d}$$

$$=\frac{(2+\beta-1)}{1+1}\vec{d}=\frac{1+\beta}{2}(-1,1,0)$$

point on the line closest to B is $A+\text{proj}_{\vec{d}}(\vec{v})=$

$$=(3,1,1)+\frac{1+\beta}{2}(-1,1,0)=\left(\frac{5-\beta}{2},\frac{3+\beta}{2},1\right)$$

shortest distance is $\|\vec{BP}\|=\left\|(1,\beta,7)-\left(\frac{5-\beta}{2},\frac{3+\beta}{2},1\right)\right\|$

$$=\left\|\left(\frac{\beta-3}{2},\frac{\beta-3}{2},6\right)\right\|=\sqrt{\frac{1}{2}(\beta-3)^2+36}$$

My answer: ____________________

3. (a) (3 points) Determine an equation of the plane containing the lines

$$[3 \quad 1 \quad 0]^T + t[1 \quad 2 \quad 3]^T \quad \text{and} \quad [2 \quad 4 \quad 22]^T + t[-1 \quad -1 \quad 2]^T.$$

The normal $\vec{n}$ to the two lines is $\vec{n} = (1,2,3) \times (-1,-1,2)$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 2 & 3 \\ -1 & -1 & 2 \end{vmatrix} = \left(\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} \right) = (7,-5,1)$$

The plane with normal $\vec{n} = (7,-5,1)$ through the point $(3,1,0)$ [which lies on the 1st line] is

$$0 = \vec{n} \cdot (x-3, y-1, z) = (7,-5,1) \cdot (x-3, y-1, z)$$
$$= 7x - 5y + z - 16$$

My answer: $7x - 5y + z = 16$

(b) (2 points) Find the equation of the plane passing through the point $P(1,-2,1)$ and parallel to the plane $3x - 2y + z = 5$.

normal of the plane is $\vec{n} = (3,-2,1)$

equation of the plane is therefore

$$0 = \vec{n} \cdot ((x,y,z) - (1,-2,1)) = (3,-2,1) \cdot (x-1, y+2, z-1)$$
$$= 3x - 3 - 2y - 4 + z - 1 = 3x - 2y + z - 8$$

My answer: $3x - 2y + z = 8$

4. (a) (3 points) Find the volume of the parallelepiped determined by the vectors $A(-1,0,\gamma)$, $B(2,1,1)$ and $C(-1,-1,3)$, where γ is the third last digit of your student number.

$$\vec{B}\times\vec{C} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -1 & -1 & 3 \end{vmatrix} = \left(\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} \right) = (4,-7,-1)$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (-1,0,\gamma)\cdot(4,-7,-1) = -4-\gamma$$

The volume is $|\vec{A}\cdot(\vec{B}\times\vec{C})| = |-4-\gamma| = 4+\gamma$

My answer: $4+\gamma$

(b) (2 points) Find the area of the triangle with vertices $P(1,2,-3)$, $Q(1,1,0)$ and $R(2,3,0)$.

$$\vec{PQ} = (1,1,0)-(1,2,-3) = (0,-1,3)$$

$$\vec{PR} = (2,3,0)-(1,2,-3) = (1,1,3)$$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ 0 & -1 & 3 \\ 1 & 1 & 3 \end{vmatrix} = \left(\begin{vmatrix} -1 & 3 \\ 1 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} \right) =$$

$$= (-6,3,1)$$

$$\|\vec{PQ}\times\vec{PR}\| = \sqrt{36+9+1} = \sqrt{46}$$

area of triangle is $\frac{1}{2}\|\vec{PQ}\times\vec{PR}\| = \frac{1}{2}\sqrt{46}$

My answer: $\frac{1}{2}\sqrt{46}$