University of Ottawa

Department of Mathematics and Statistics

MAT 1341 B: Introduction to Linear Algebra

Instructor: Erhard Neher

Final Exam, Dec. 7, 2007

Family Name: ______________________

First Name: ______________________

Student number: ______________________

Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate that you have done this.
- For all questions you must show your work to obtain the points. Simply writing the correct answer will earn you 0.
- Please write legibly and argue logically: You must convince me that you know why your solution is correct.
- The exam has 14 questions; questions 15 and 16 are bonus questions. There is an extra page at the end of the exam which you can use. Please do not remove this page.
- No books or notes are allowed. **Calculators are not permitted.**

Good luck!

Quest.	1 – 4	5 – 6	7 – 8	9 – 10	11 – 12	13 – 14	15 – 16	Total
max.	8	6	7	11	14	14	4(bonus)	60
score								

1. (2 points) Suppose a directed graph has adjacency matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

How many paths of length 2 are there from vertex 4 to vertex 2?

My answer:_______________

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My answer:_______________

Answer: By Theorem 5 in section 1.4, the number of paths of length 2 from vertex i to vertex j is the (i,j)-entry of A^2, that is the dot product $R_i \cdot C_j$, where R_i is the i-th row and C_j is the j-th column of the adjacency matrix A of the graph. Thus, for the first question we have to calculate

$$R_4 \cdot C_2 = [1 \quad 0 \quad 0 \quad 0] \cdot [0 \quad 0 \quad 1 \quad 0] = 0$$

and for the second question we need to calculate

$$R_2 \cdot C_4 = [0 \quad 0 \quad 1 \quad 0] \cdot [0 \quad 0 \quad 1 \quad 0] = 1$$

Hence, the answers are 0 and 1.

2. (2 points) Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Find $(AB)^{-1}(BA^T)^T$.

Answer: From the rules of matrix multiplication we get

$$(AB)^{-1}(BA^T)^T = B^{-1}A^{-1}(A^T)^T B^T = B^{-1}A^{-1}AB^T = B^{-1}B^T.$$

Since $\det(B) = -3$, we obtain

$$B^{-1}B^T = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & -5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

My answer:______________________________

3. (2 points) Let A, B and C be 3×3 matrices with $\det(A) = 3$, $\det(B) = 5$ and $\det(C) = 6$. Find the determinant of the 3×3 matrix M satisfying $AMB = C$.

Answer: By the product rule for determinants, we have

$$\det(AMB) = \det(A)\det(M)\det(B) = 3 \cdot 5 \det(M) = 15 \det(M).$$

But $AMB = C$, whence $\det(AMB) = \det(C) = 6$. Thus, $15 \det(M) = 6$, and therefore $\det(M) = \frac{6}{15} = \frac{2}{5}$.

My answer:________________

4. (2 points) Find the inverse of

$$A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$$

Answer: The determinant of A is $2 + i^2 = 1$. Hence the inverse of A is

$$A^{-1} = \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}$$

My answer:________________________________

5. (3 points) The dimension of the subspace of $\mathbb{R}^5$ spanned by the vectors

$$[1 \quad 0 \quad 3 \quad 1 \quad 1]^T, \quad [1 \quad -1 \quad 7 \quad -1 \quad 0]^T, \quad [2 \quad 1 \quad 2 \quad 4 \quad 3]^T \quad \text{and} \quad [5 \quad 1 \quad 11 \quad 7 \quad 6]^T$$

is

Answer: The subspace is the same as the row space of the matrix A below. We can therefore apply Theorem 2 in section 4.4 to find its dimension:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 1 & -1 & 7 & -1 & 0 \\ 2 & 1 & 2 & 4 & 3 \\ 5 & 1 & 11 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & -1 & 4 & -2 & -1 \\ 0 & 1 & -4 & 2 & 1 \\ 0 & 1 & -4 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & -4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank of A is 2, which is then also the dimension of the subspace we are looking for.

My answer:_______________

6. (3 points) Suppose that $T\colon \mathbb{M}_{2,2} \to \mathbb{P}_2$ is a linear transformation satisfying

$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = x, \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = x^2, \quad T\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = 1 - x, \quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right) = 5 - x^2.$$

Find $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$.

Answer: We write $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as a linear combination of the four matrices, for which we know the value of T:

$$x_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

By comparing coefficients, this yields a linear system

$$x_1 = 1, \quad x_1 + x_2 = 0, \quad x_3 = 0, \quad x_2 + x_3 - x_4 = 0,$$

whose solution is $x_1 = 1$, $x_3 = 0$, $x_2 = -1 = x_4$. Thus,

$$\begin{aligned} T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right) \\ &= x - x^2 - (5 - x^2) = -5 + x. \end{aligned}$$

My answer:_____________________________

7. (3 points) For an $n \times n$ matrix A answer the questions (ii), (iii) and (iv). As an example, I have given the answer for (i).

(i) State a condition on the columns of A which is equivalent to the condition rank $(A) < n$.

Answer: The columns of A are linearly dependent

(ii) State a condition on the determinant of A which is equivalent to the condition rank $(A) < n$.

Answer: $\det(A) = 0$

My answer:______________________________

(iii) State a condition regarding the homogeneous linear system $AX = 0$ which is equivalent to the condition rank $(A) < n$.

Answer: The homogeneous linear system $AX = 0$ has a non-trivial solution.

My answer:______________________________

(iv) State a condition on invertibility of A which is equivalent to the condition rank $(A) < n$.

Answer: A is not invertible.

My answer:______________________________

8. (4 points) Complete the formulas below:

(i) If A is an $m \times n$ matrix, then $n - \dim(nullA) - \text{rank}\,(A) =$

Answer: $= 0$

My answer:_______________

(ii) If A is an $m \times n$ matrix, then $\dim(colA) - \text{rank}\,(A) =$

Answer: $= 0$.

My answer:_______________

(iii) $\dim \mathbb{P}_4 =$

Answer: $= 5$.

My answer:_______________

(iv) $\dim \mathbb{M}_{2,4} =$

Answer: $= 8$.

My answer:_______________

9. (4 points) Give the general solution of the system of first order differential equations

$$\begin{array}{rcrcl} 2f_1 & + & 4f_2 & = & f_1' \\ & & -3f_2 & = & f_2' \end{array}$$

(Partial marks will be given for diagonalizing the coefficient matrix of the system.)

Answer: The system is equivalent to the system $Af = f'$ for

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$

Since A is upper-triangular, the eigenvalues of A are the diagonal elements 2 and -3. Since A has 2 distinct eigenvectors, it is diagonalizable. We find a basis of eigenvectors:

For $\lambda = 2$ we get

$$2I_2 - A = \begin{bmatrix} 0 & -4 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence a basic eigenvector for $\lambda = 2$ is

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We proceed similarly for $\lambda = -3$:

$$-3I_3 - A = \begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 4 \\ 0 & 0 \end{bmatrix}$$

The corresponding equation is $5x_1 + 4x_2 = 0$. Hence

$$X_2 = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

is a basic eigenvector for $\lambda = -3$. By Theorem 1 in section 2.8, the general solution of the system is

$$f(x) = c_1 \, e^{2x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \, e^{-3x} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Marking: 3 points for finding e-values and a basis of e=vectors; 1 point for the general solution

10. (7 points) Determine the values of a for which the linear system

$$\begin{array}{rcrcl} x & + & ay & = & 1 \\ ax & + & 4y & = & 2 \end{array}$$

has (i) no solution, (ii) a unique solution, (iii) infinitely many solutions. In case (iii) determine all solutions.

Answer: The augmented matrix of the system is

$$\begin{bmatrix} 1 & a & 1 \\ a & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & a & 1 \\ 0 & 4-a^2 & 2-a \end{bmatrix}$$

Since $4 - a^2 = (2-a)(2+a)$ we get the following cases:

(i) No solution: This is the case when we have a row of type $[0 \quad 0 \quad 1]$ in a row-echelon form of the augmented matrix. Hence, this is the case if and only if $4 - a^2 = 0$ and $a \neq 2$, which is equivalent to $a = -2$.

(ii) A unique solution: This is the case when the augmented matrix has rank 2, but we do not have a row of type $[0 \quad 0 \quad 1]$ in a row-echelon form of A. Hence, this is the case exactly when $4 - a^2 \neq 0$, i.e., $a \neq 2$ and $a \neq -2$.

(iii) Infinitely many solutions: The rank of the augmented matrix is 1, i.e., $2 - a = 0 = 4 - a^2$, which is equivalent to $a = 2$. In this case, a row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the corresponding linear system becomes $x + 2y = 1$, whose solutions are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-2t \\ t \end{bmatrix}$$

where t is a free parameter.

Marking: 3 points for row-reduction; 1 point each for finding the cases (ii)–(iii); 1 point for solution in case (iii)

11. (6 points) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

is $x(x-2)^2$. (You do not need to show this.)

(a) (4 points) For each eigenvalue of A find a basis of the corresponding eigenspace.

(b) (2 points) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If no, explain why not.

Answer: Since the eigenvalues are the roots of the characteristic polynomials, A has eigenvalues 2 and 0. For each eigenvalue λ, the corresponding eigenspace is the set of solutions of the homogenous linear system $(\lambda I_3 - A)$.

$\lambda = 2$: We row-reduce

$$2I_3 - A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding linear system is $x + y - z = 0$ or $x = -y + z$. The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s+t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where s and t are free parameters. Hence the eigenspace $E_2(A)$ has

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as a basis, and thus $\dim E_2(A) = 2$.

$\lambda = 0$: We row-reduce

$$0I_3 - A = \begin{bmatrix} -3 & -1 & 1 \\ 1 & -1 & -1 \\ -2 & -2 & 0 \end{bmatrix} \overset{(1)}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ -3 & -1 & 1 \\ -2 & -2 & 0 \end{bmatrix} \overset{(2)}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -4 & -2 \\ 0 & -4 & -2 \end{bmatrix} \overset{(3)}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{(4)}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \overset{(5)}{\sim} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Explanations: (1): exchange rows 1 and 2; (2): replace row R2 by R2+3R1 and row R3 by R3 +2R1; (3): replace row R3 by R3-R2 etc. Hence the corresponding linear system is $x = (1/2)z$ and $y = -(1/2)z$. Its general solution is

$$\begin{bmatrix} (1/2)s \\ -(1/2)s \\ s \end{bmatrix} = s \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

where s is a free parameter. Thus, a basis of the eigenspace $E_0(A)$ is

$$\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

(b) Since the dimensions of the eigenspaces add up to 3, the matrix is diagonalizable. Matrixes P and D as required in (b) are

$$P = \begin{bmatrix} -1 & 1 & -1/2 \\ 1 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that P and D are not unique.

Marking: Correct calculations, but basis not stated: -1; eigenspace $\{0\}$: -2; in (b) it is sufficient just to give the matrices P and D;

Wrong answers in (b): A is not diagonalizable since its determinant os 0; A is not diagonalizable since it has only 2 eigenvalues; A is diagonalizable since it has 3 eigenvectors; A is not diagonalizable since the elements of the matrix are linearly independent;

12. (8 points) Let

$$\begin{bmatrix} 2 & 6 & 10 & 4 \\ 1 & 3 & 6 & 1 \\ 3 & 9 & 14 & 7 \end{bmatrix}.$$

(a) (4 points) Find the reduced row-echelon form of A.
(b) (1 points) Give a basis of the row space $row(A)$ of A.
(c) (1 points) Give a basis of the column space $col(A)$ of A.
(d) (2 points) Give a basis of the null space $null(A)$.

Answer: (a)

$$A \sim \begin{bmatrix} 1 & 3 & 6 & 1 \\ 2 & 6 & 10 & 4 \\ 3 & 9 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{ref}(A).$$

(b) A basis of the row space is given by the non-zero rows of '$('A)$ (Thm. 1 in section 4.4):

$$[1 \quad 3 \quad 0 \quad 7], \quad [0 \quad 0 \quad 1 \quad -1].$$

(c) By Theorem 2 in section 4.4, a basis of the column space of A is given by those columns of A corresponding to the columns of ref(A) with a leading 1. These are the columns 1 and 3. Hence a basis of $col(A)$ is

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 10 \\ 6 \\ 14 \end{bmatrix}.$$

(d) The null space of A is the set of solutions of the linear system $AX = 0$, which is equivalent to ref$(A)X = 0$. The latter system has the form

$$\begin{array}{ccccccccc} x_1 & + & 3x_2 & & & + & 7x_4 & = & 0 \\ & & & & x_3 & - & x_4 & = & 0 \end{array}$$

We see that x_2 and x_4 are free variables, and that the general solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s - 7t \\ s \\ t \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where s and t are free parameters. Hence a basis of $null(A)$ is

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Marking: Basis = spanning set: errors in the ref: -1 or -2, depending on error;

13. (7 points) Let

$$U = \{A \in \mathbb{M}_{3,3} : A + A^T = 0\}.$$

For example, the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

lies in U, since if we add the matrix and its transpose we obtain the zero matrix.

(a) (3 points) Show that U is a subspace of $\mathbb{M}_{3,3}$.

(b) (3 points) Find a basis of U.

(c) (1 point) Determine the dimension of U.

Answer: (a) We can apply the subspace test.

(1) The zero matrix satisfies $0^T + 0 = 0 + 0 = 0$ so 0 is in U.

(2) Given A, B in U, we have that $A + A^T = 0$ and $B + B^T = 0$. Thus $(A+B) + (A+B)^T = A + B + A^T + B^T = (A + A^T) + (B + B^T) = 0 + 0 = 0$ so $A + B$ is also in U.

(3) Given A in U and c a scalar, we have $(cA) + (cA^T) = cA + cA^T = c(A + A^T) = c0 = 0$ so cA is in U.

Thus U satisfies all three tests and is a subspace of $\mathbb{M}_{3,3}$.

(b) If

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

then

$$A + A^T = \begin{bmatrix} 2a & b+d & c+g \\ b+d & 2e & f+h \\ c+g & f+h & 2i \end{bmatrix}$$

So we deduce that A is in U if and only if

$$a = e = i = 0, b = -d, c = -g, f = -h$$

so

$$U = \left\{ \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix} : b, c, f \in \mathbb{R} \right\}$$

In particular, this says that any matrix in U has the form

$$b \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

so U is the span of the three matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Furthermore, these are linearly independent (since the only linear combination that can make the zero matrix is the trivial one). Hence this is a basis for U.

(c) Since there are 3 vectors in a basis, this is a 3 dimensional subspace of $\mathbb{M}_{3,3}$.

Marking: (b) Just giving a basis: 1/3; proving spanning without verifying linear independence 2/3.

Mistakes: Row-reducing the example matrix to find a basis of U; calculating the characteristic polynomial of the given example matrix.

Some wrong answers: "U is a subspace since U must fit within the subspace of $\mathbb{M}_{3,3}$ since otherwise dimensions will be left out."

14. (7 points) Let U be the span of the following set of vectors in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(i) (2 points) Show that these vectors are orthogonal.
(ii) (1 points) What is $\dim(U)$?
(iii) (1 points) What is $\dim(U^\perp)$?
(iv) (3 points) Calculate the orthogonal projection of the vector $v = [1 \quad 2 \quad 3 \quad 1]^T$ onto U.

Answer: (i) We have that

$$[1 \quad -1 \quad 2 \quad 0]^T \cdot [3 \quad 3 \quad 0 \quad 1]^T = 3 - 3 + 0 + 0 = 0$$

$$[1 \quad -1 \quad 2 \quad 0]^T \cdot [-1 \quad 1 \quad 1 \quad 0]^T = -1 - 1 + 2 + 0 = 0$$

and

$$[3 \quad 3 \quad 0 \quad 1]^T \cdot [-1 \quad 1 \quad 1 \quad 0]^T = -3 + 3 + 0 + 0 = 0$$

Since all pairwise dot products are zero, this is an orthogonal set of vectors in $\mathbb{R}^4$.

(ii) Since the set is orthogonal, it is linearly independent. Therefore it is a basis for its span U. Hence $\dim(U) = 3$.

(iii) Since $\dim(U) + \dim(U^\perp) = \dim(\mathbb{R}^4)$ and $\dim(U) = 3$ we deduce $\dim(U^\perp) = 4 - 3 = 1$.

(iv) Since we have an orthogonal basis (call it $\{X_1, X_2, X_3\}$), we can apply the projection formula:

$$\text{proj}_U(X) = \frac{X \cdot X_1}{\|X_1\|^2} X_1 + \frac{X \cdot X_2}{\|X_2\|^2} X_2 + \frac{X \cdot X_3}{\|X_3\|^2} X_3$$

Here $X = [1 \quad 2 \quad 3 \quad 1]^T$, so

$$X \cdot X_1 = 5, X \cdot X_2 = 10, X \cdot X_3 = 4$$

and

$$\|X_1\|^2 = 6, \|X_2\|^2 = 19, \|X_3\|^2 = 3.$$

So

$$\text{proj}_U(X) = \frac{5}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} + \frac{10}{19} \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{95}{114} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} + \frac{60}{114} \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \frac{152}{114} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{114} \begin{bmatrix} 123 \\ 237 \\ 342 \\ 60 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 41 \\ 79 \\ 114 \\ 20 \end{bmatrix}$$

We can check that this is correct by verifying that $X - proj_U(X) = \frac{1}{114}[-9 \quad -9 \quad 0 \quad 54]^T$ is orthogonal to U.

Another approach to (iv): Let's find a basis for $U^\perp$, which is one-dimensional. If $U = row(A)$ then $U^\perp = Null(A)$ so:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 3 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 6 & -6 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and so a basis for the nullspace $U^\perp$ is $Y = [-1 \quad -1 \quad 0 \quad 6]^T$. Therefore

$$\text{proj}_{U^\perp} X = \frac{X \cdot Y}{\|Y\|^2} Y = \frac{3}{38} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 6 \end{bmatrix}$$

and so

$$\text{proj}_U(X) = X - \text{proj}_{U^\perp}(X) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} - \frac{3}{38} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 6 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 41 \\ 79 \\ 114 \\ 20 \end{bmatrix}$$

Marking: (a) -1 if orthogonality only checked for the first vectors; (d) -1 if some calculation error but method correct; full marks if the projection vector was given as the correct linear comb of the orthogonal basis, but was not simplified; formula wrong: 1/3.

Some mistakes: Calculating the vector approximately, i.e., replacing the fractions by integers!!! Using the formula for the component of X in $U^\perp$, instead of the projection on U.

Not really a mistake but too complicated: Applying Gram-Schmidt to check orthogonality; determining the dimension of U by row-reducing the matrix whose rows (or columns) are the given three vectors.

15. (**2 bonus points**). The following statement is a theorem that we have seen in class, except that one of the hypotheses is missing. Write down the missing hypothesis.

Let $v_1, \dots, v_n$ be vectors in a vector space V. Suppose that

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If one of the conditions

(i) $\{v_1, \dots, v_n\}$ *is a linearly independent set*

(ii) $\{v_1, \dots, v_n\}$ *is a spanning set for* V

is satisfied, then both of them are satisfied.

Answer: Missing hypothesis: $\dim(V) = n$

Marking: Some wrong answers: V is diagonalizable; V is invertible; the v_i are orthogonal;

16. (**2 bonus points**) Complete the following definition:

"Vectors $v_1, \dots, v_n$ of a vector space V are called *linearly dependent* if there are real numbers $a_1, \dots, a_n \in \mathbb{R}$ such that

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Answer: ... such that $a_1v_1 + \cdots + a_nv_n = 0$ and not all of the a_i are zero.

Marking: Some wrong answers: $a_1v_1 + \cdots + a_nv_n \neq 0$; "all vectors $v_1, \dots, v_n$ can be divided to form a single vector"; giving the definition of linear independence instead of linear dependence (1/2).