University of Ottawa

Department of Mathematics and Statistics

MAT 1341 B: Introduction to Linear Algebra

Instructor: Erhard Neher

Final Exam, Dec. 7, 2007

Family Name: ______________________

First Name: ______________________

Student number: ______________________

Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate that you have done this.
- For all questions you must show your work to obtain the points. Simply writing the correct answer will earn you 0.
- Please write legibly and argue logically: You must convince me that you know why your solution is correct.
- The exam has 14 questions; questions 15 + 16 are bonus questions. There is an extra page at the end of the exam which you can use. Please do not remove this page.
- No books or notes are allowed. **Calculators are not permitted.**

Good luck!

Quest.	1 – 4	5 – 6	7 – 8	9 – 10	11 – 12	13 – 14	15 – 16	Total
max.	8	6	7	11	14	14	4(bonus)	60
score								

1. (2 points) Suppose a directed graph has adjacency matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

How many paths of length 2 are there from vertex 4 to vertex 2?

My answer:________________

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My answer:________________

2. (2 points) Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Find $(AB)^{-1}(BA^T)^T$.

My answer:________________________________

3. (2 points) Let A, B and C be 3×3 matrices with $\det(A) = 3$, $\det B = 5$ and $\det(C) = 6$. Find the determinant of the 3×3 matrix M satisfying $AMB = C$.

My answer:________________

4. (2 points) Find the inverse of

$$A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$$

My answer:______________________________

5. (3 points) The dimension of the subspace of $\mathbb{R}^5$ spanned by the vectors

$$[1 \quad 0 \quad 3 \quad 1 \quad 1]^T, \quad [1 \quad -1 \quad 7 \quad -1 \quad 0]^T, \quad [2 \quad 1 \quad 2 \quad 4 \quad 3]^T \quad \text{and} \quad [5 \quad 1 \quad 11 \quad 7 \quad 6]^T$$

is

My answer:________________

6. (3 points) Suppose that $T\colon \mathbb{M}_{2,2} \to \mathbb{P}_2$ is a linear transformation satisfying

$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = x, \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = x^2, \quad T\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = 1 - x, T\left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right) = 5 - x^2.$$

Find $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$.

My answer:______________________________

7. (3 points) For an $n \times n$ matrix A answer the questions (ii), (iii) and (iv). As an example, I have given the answer for (i).

(i) State a condition on the columns of A which is equivalent to the condition $\text{rank}\,(A) < n$.

Answer: The columns of A are linearly dependent

(ii) State a condition on the determinant of A which is equivalent to the condition $\text{rank}\,(A) < n$.

My answer:________________________________

(iii) State a condition regarding the homogeneous linear system $AX = 0$ which is equivalent to the condition $\text{rank}\,(A) < n$.

My answer:________________________________

(iv) State a condition on invertibility of A which is equivalent to the condition $\text{rank}\,(A) < n$.

My answer:________________________________

8. (4 points) Complete the formulas below:

(i) If A is an $m \times n$ matrix, then $n - \dim(nullA) - \text{rank}\,(A) =$

My answer:________________

(ii) If A is an $m \times n$ matrix, then $\dim(colA) - \text{rank}\,(A) =$

My answer:________________

(iii) $\dim \mathbb{P}_4 =$

My answer:________________

(iv) $\dim \mathbb{M}_{2,4} =$

My answer:________________

9. (4 points) Give the general solution of the system of first order differential equations

$$\begin{array}{rcrcl} 2f_1 & + & 4f_2 & = & f_1' \\ & & -3f_2 & = & f_2' \end{array}$$

(Partial marks will be given for diagonalizing the coefficient matrix of the system.)

10. (7 points) Determine the values of a for which the linear system

$$\begin{array}{rcrcl} x & + & ay & = & 1 \\ ax & + & 4y & = & 2 \end{array}$$

has (i) no solution, (ii) a unique solution, (iii) infinitely many solutions. In case (iii) determine all solutions.

11. (6 points) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

is $x(x-2)^2$. (You do not need to show this.)

(a) (4 points) For each eigenvalue of A find a basis of the corresponding eigenspace.

(b) (2 points) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If no, explain why not.

12. (8 points) Let

$$\begin{bmatrix} 2 & 6 & 10 & 4 \\ 1 & 3 & 6 & 1 \\ 3 & 9 & 14 & 7 \end{bmatrix}.$$

(a) Find the reduced row-echelon form of A.
(b) Give a basis of the row space $row(A)$ of A.
(c) Give a basis of the column space $col(A)$ of A.
(d) Give a basis of the null space $null(A)$.

13. (7 points) Let

$$U = \{A \in \mathbb{M}_{3,3} : A + A^T = 0\}.$$

For example, the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

lies in U, since if we add the matrix and its transpose we obtain the zero matrix.

(a) (3 points) Show that U is a subspace of $\mathbb{M}_{3,3}$.

(b) (3 points) Find a basis of U.

(c) (1 point) Determine the dimension of U.

14. (7 points) Let U be the span of the following set of vectors in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(i) (2 points) Show that these vectors are orthogonal.
(ii) (1 points) What is $\dim(U)$?
(iii) (1 points) What is $\dim(U^\perp)$?
(iv) (3 points) Calculate the orthogonal projection of the vector $v = [1 \quad 2 \quad 3 \quad 1]^T$ onto U.

15. (**2 bonus points**). The following statement is a theorem that we have seen in class, except that one of the hypotheses is missing. Write down the missing hypothesis.

Let $v_1, \ldots, v_n$ be vectors in a vector space V. Suppose that

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If one of the conditions

(i) $\{v_1, \ldots, v_n\}$ *is a linearly independent set*
(ii) $\{v_1, \ldots, v_n\}$ *is a spanning set for* V

is satisfied, then both of them are satisfied.

16. (**2 bonus points**) Complete the following definition:

"Vectors $v_1, \ldots, v_n$ of a vector space V are called *linearly dependent* if there are real numbers $a_1, \ldots, a_n \in \mathbb{R}$ such that

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(extra page)