2. Derivations

Derivation are "infinitesimal automorphisms", and as such have a theory formally analogous to that of automorphisms. Again derivations are difficult to construct. The inner derivations are built up from commutator and associator maps in such a way that the indicator lies in the nucleus. The basic inner derivations are the standard inner derivations D_z and $D_{x,y}$ for nuclear z and arbitrary x,y, although these are insufficient in characteristic 3 situations. We show that standard inner derivations are infinitesimal generators of the standard inner automorphisms.

Recall that a **derivation** in any linear algebra is an endomorphism D of A which satisfies the "product rule" for derivatives

(2.1) $$D(x\cdot y) = D(x)\cdot y + x\cdot D(y).$$

The fact that one applies D to a product by applying it to the factors one at a time and summing has, as consequences, rules such as

(2.2) $$D[x,y] = [Dx,y] + [x,Dy]$$

(2.3) $$D[x,y,z] = [Dx,y,z] + [x,Dy,z] + [x,y,Dz]$$

(2.4) $$D(x^2) = x\circ D(x)$$

(2.5) $$D(U_x y) = U_x Dy + U_{x,Dx} y.$$

Derivations kill units

(2.6) $$D(1) = 0$$

because by (2.4) $D(1) = D(1^2) = 1\circ D(1) = 2D(1)$. The rule for differentiating an inverse in an alternative algebra

$$D(x^{-1}) = -U_{x^{-1}} D(x)$$

generalizes the "quotient rule" $(\frac{1}{f})' = -\frac{f'}{f^2}$ for derivatives. Simply note $0 = D(1) = D(xx^{-1}) = D(x)x^{-1} + xD(x^{-1})$, so $xD(x^{-1}) = -D(x)x^{-1}$ and (by the Inverse Condition $L_x{}^{-1} = L_{x^{-1}}$) $D(x^{-1}) = -x^{-1}D(x)x^{-1}$.

As an application of (2.2) and (2.3), note that a derivation preserves nucleus and center in any linear algebra.

$$D(N(A)) \subset N(A), \qquad D(C(A)) \subset C(A)$$

because for $n \in N(A)$ and $a,b \in A$ we have by (2.3) $[Dn,a,b] = D[n,a,b] - [n,Da,b] - [n,a,Db] = 0$, similarly for middle and right nuclearity of Dn, and if in addition $n \in C(A)$ then $[Dn,a] = D[n,a] - [n,Da] = 0$ by (2.2).

Still in the general case, the set of derivations forms a Lie subalgebra of $End(A)^-$: from the linearity of (2.1) in the variable D it is clear that αD and $D_1 + D_2$ are derivations if D, D_1, D_2 are, while for the Lie bracket or commutator $[D_1,D_2] = D_1D_2 - D_2D_1$ we interchange 1 and 2 and subtract in $D_1D_2(xy) = D_1\{D_2(x)y + xD_2(y)\} = D_1D_2(x)y + D_2(x)D_1(y) + D_1(x)D_2(y) + xD_1D_2(y)$ to get $[D_1,D_2](xy) = [D_1,D_2](x)y + x[D_1,D_2](y)$. This Lie algebra is called the **derivation algebra** $Der(A)$ of the algebra A.

The **Lie multiplication algebra** $LM(A)$ of any linear algebra A is the Lie algebra of linear transformations on A generated by all left and right multiplications L_x, R_y for $x,y \in A$ (the smallest subspace of $End_\Phi A$ containing the L_x and R_y and closed under the Lie bracket). A derivation of A which belongs to $LM(A)$ (i.e. can be built out of the L_x and R_y by means of Lie brackets) is called an **inner derivation**, and the space of all such is denoted

$$Inder(A) = Der(A) \cap LM(A).$$

In operator terms the derivation condition (2.1) becomes

(2.7) $$[D, L_x] = L_{D(x)}$$
$$[D, R_y] = R_{D(y)}$$

which shows the generators (and hence all of LM(A)) are invariant under bracketing by derivations. Thus the inner derivations form a Lie ideal,

$$\mathrm{Inder}(A) \triangleleft \mathrm{Der}(A).$$

The inner derivations are those which result from the interaction of multiplication operators. We are especially interested in inner derivations of free algebras in a variety; by (2.7) these correspond to certain operator indentities in the variety. We now turn to the question of describing these inner derivations more concretely in the alternative case.

Criteria for Inner Derivations

From now on we consider only alternative algebras. The most important derivations in an associative algebra are the **adjoint** or **commutator** maps

(2.8) $$D_z : x \to [z,x] \qquad (D_z = L_z - R_z).$$

In an alternative algebra these maps D_z need not be derivations; indeed, recall the associator-commutator formula II.2.10, which we write as

(2.9) $$[z,xy] = [z,x]y + x[z,y] - 3[z,x,y]$$

Thus $D_z(xy) - D_z(x)y - xD_z(y) = -3[z,x,y]$ measures how far D_z is from being a derivation. This is an easy way to remember the formula, so we call it the **commutator derivation formula.**

The condition that D_z be a derivation is precisely that the associator vanishes for all x and y, i.e. the **indicator** 3z is nuclear.

2.10 (Commutator Derivation Condition) The map $D_z = L_z - R_z$ is a derivation of an alternative algebra A iff the indicator 3z lies in the nucleus N(A). ∎

In particular, if 3A = 0 or A = N(A) is associative then <u>all</u> D_z are derivations.

Another condidate for a derivation is the **associator** map

(2.11) $A_{x,y}: z \to [x,y,z] \qquad (A_{x,y} = L_{xy} - L_x L_y = [L_x, R_y])$

where $A_{x,y}(z) = -[x,z,y] = -\{R_y L_x - L_x R_y\}z = [L_x, R_y]z$.

The relevant formula here is the associator-commutator formula II.2.12, written

(2.12) $[x,y,zw] = [x,y,z]w + z[x,y,w] - [[x,y],z,w]$.

Again, $A_{x,y}(zw) - A_{x,y}(z)w - zA_{x,y}(w) = -[[x,y],z,w]$ measures how far $A_{x,y}$ is from being a derivation. Again we call $[x,y]$ the **indicator** of the associator map, since it indicates whether $A_{x,y}$ is a derivation.

2.13 (Associator Derivation Condition) The map $A_{x,y} = [L_x, R_y]$ is a derivation of an alternative algebra A iff the indicator $[x,y]$ lies in the nucleus N(A). A sum $\Sigma A_{x_i,y_i}$ of associator maps is a derivation iff $\Sigma[x_i,y_i] \in N(A)$. ∎

If x and y commute ($[x,y] = 0$) or if A = N(A) is associative then $A_{x,y}$ is a derivation.

Although it is a little hard to remember, formula (2.12) can sometimes be as useful as our basic formulas in Section 1.3. Again, the best way to remember it is as a measure of deviation from being a derivation; we call it the **associator derivation formula**. The hard part is remembering the error term $-[[x,y],z,w]$ (the coefficient −1 is not to be confused with the −3 of the commutator derivation formula).

Even more useful than the commutator and associator derivation formulas (because there are no error terms) are the facts that D_z and $A_{z,w}$ act as derivations on all Jordan products, even if not on all alternative products (see the Jordan Derivation Formulas I.3.7-8).

2.14 Example. In a Cayley algebra $A = \mathfrak{C}(B,\mu)$ we have the following expressions for the action of the commutator and associator maps ($x,y \in B$):

$$D_x\,(a + b\ell) = [x,a] + \{b(x-\bar{x})\}\ell$$

$$D_{x\ell}\,(a + b\ell) = \mu\{\bar{b}x - \bar{x}b\} + \{x(\bar{a} - a)\}\ell$$

$$A_{x,y}\,(a + b\ell) = \{b[x,y]\}\ell$$

$$A_{x,y\ell}\,(a + b\ell) = \mu[b\bar{y},x] + \{y[x,\bar{a}]\}\ell$$

$$A_{x\ell,\bar{y}\ell}\,(a + b\ell) = \mu[yx,a] + \mu\{byx - xyb\}\ell. \quad \blacksquare$$

Thus far we have been able to construct inner derivations only from very special kinds of elements. Once more the best way to discover the form inner derivations take is to find general conditions under which an element of the Lie multiplication algebra is a derivation. We begin by finding a normal form for the elements of the Lie multiplication algebra.

2.15 (Normal Form Theorem) Any element of the Lie multiplication algebra of an alternative algebra may be written in the form

$$W = L_x + R_y + \Sigma[L_{x_i}, R_{y_i}],$$

so the Lie multiplication algebra reduces to

$$LM(A) = L_A + R_A + [L_A, R_A].$$

Proof. To see this we only need show the subspace on the right, which manifestly contains all L_x and R_y, is closed under Lie brackets. This follows because $[L_x,L_y]$ and $[R_x,R_y]$ can be expressed in terms of $[L_x,R_y]$

$$[L_x L_y]-L_{[xy]} = -2[L_x,R_y] = [R_x,R_y]+R_{[xy]}$$

and because the $[L_x,R_y]$'s are closed under bracketing with the generators L_z,R_w

$$[L_z,[L_x,R_y]] = -L_{[xyz]} + [L_{[xy]},R_z]$$

$$[R_z,[R_x,L_y]] = -R_{[xyz]} - [R_{[xy]},L_z].$$

For the first equation: $x(ya)-y(xa)+(yx-xy)a = -[x,y,a]+[y,x,a]$ $= +2[x,a,y] = [a,y,x] - [a,x,y] = (ay)x-(ax)y+a(xy-yx)$. For the second: $zA_{x,y}(a)-A_{x,y}(za) = -A_{x,y}(z)a+[[xy],z,a]$ (by 2.12)) $= -L_{[xyz]}(a) + A_{[xy],z}(a)$. The third follows by duality – in the oposite algebra R and L get switched, $[x,y]$ and $[x,y,z]$ become $[y,x] = -[x,y]$ and $-[z,y,x] = +[x,y,z]$ respectively. ∎

Once we have a way of representing the elements of the Lie multiplication algebra, we seek a criterion for when such an element is a derivation. The **indicator** ind(D) of a sum

$$D = D_x + \Sigma A_{x_i,y_i}$$

of commutator and associator maps is just the sum $3x + \Sigma[x_i,y_i]$ of the respective indicators. Note the indicator is not intrinsically determined by D, but rather depends on a particular representation of D; for example, the zero derivation can be written as $D = 0$ (with indicator 0) or as $D = D_1$ (with indicator 3).

2.16 (Inner Derivation Criterion) An element $D = L_x + R_y + \Sigma[L_{x_i},R_{y_i}]$ of the Lie multiplication algebra is a derivation of a unital alternative algebra A iff $y = -x$ and the indicator ind(D) $= 3x + \Sigma[x_i,y_i]$ lies in the nucleus N(A). Thus all inner derivations are built up from commutator and associator maps, $D = D_x + \Sigma A_{x_i,y_i}$.

Proof. Since a derivation kills 1 by (2.6), and all $[L_{x_i}, R_{y_i}]1 = [x_i, y_i, 1] = 0$ kill 1 automatically, for D to be a derivation it is clearly necessary that $0 = D(1) = \{L_x + R_y\}1 = x+y$. In this case $D = D_x + \Sigma A_{x_i,y_i}$ has

$$\begin{aligned} &D(zw)-D(z)w-zD(w) \\ &= \{D_x(zw)-D_x(z)w-zD_x(w)\} + \Sigma\{A_{x_i,y_i}(zw)-A_{x_i,y_i}(z)w-zA_{x_i,y_i}(w)\} \\ &= -3[x,z,w] - \Sigma[[x_i,y_i],z,w] \qquad \text{(by (2.10), (2.13))} \\ &= -[3x+\Sigma[x_i,y_i],z,w], \end{aligned}$$

which vanishes for all z,w iff $3x+\Sigma[x_i,y_i] \in N(A)$. ■

Standard Inner Derivations

Once we have a criterion for when an element of the multiplication algebra is a derivation, we can construct inner derivations. Since $3[x,y] - [3x,y] = 0$ we have as an immediate consequence of the Inner Derivation Criterion

2.17 (Standard Inner Derivation Theorem). For any elements x,y in an alternative algebra A the operator

$$D_{x,y} = D_{[x,y]} - 3A_{x,y} = L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y]$$

is an inner derivation with indicator zero. ■

A finite sum $D = D_z + \Sigma D_{x_i,y_i}$ for nuclear z will be called a **standard inner derivation**. The standard inner derivations form a subspace Stander (A) of the space of all inner derivations. When the characteristic is 3 the standard inner derivations reduce to commutator derivations $D_{z + \Sigma[x_i,y_i]}$, <u>but in characteristic ≠ 3 situations the standard inner derivations are precisely all the inner derivations</u> because $D_x + \Sigma A_{x_i,y_i} - \frac{1}{3} D(3x + \Sigma[x_i,y_i]) - \frac{1}{3}(D_{[x_i,y_i]} - 3A_{x_i,y_i}) = \frac{1}{3} D_z - \frac{1}{3} \Sigma D_{x_i,y_i}$ for $z = 3x + \Sigma[x_i,y_i] \in N(A)$:

(2.18) Inder(A) = Stander(A) when $\frac{1}{3} \in \Phi$: $D_x + \Sigma A_{x_i,y_i} = \frac{1}{3}\{D_z - \Sigma D_{x_i,y_i}\}$.

Thus in characteristic $\neq 3$ we are justified in restricting our attention to standard derivations.

The operator conditions (2.9)

$$[D,L_x] = L_{D(x)},\quad [D,R_y] = R_{D(y)}$$

for a derivation lead immediately to

$$[D,D_x] = D_{D(x)}$$

$$[D,A_{x,y}] = A_{D(x),y} + A_{x,D(y)}$$

$$[D,D_{x,y}] = D_{D(x),y} + D_{x,D(y)}$$

$$[D,D_x + \Sigma A_{x_i,y_i}] = D_{D(x)} + \Sigma A_{D(x_i),y_i} + \Sigma A_{x_i,D(y_i)}$$

$$[D,D_z + \Sigma D_{x_i,y_i}] = D_{D(z)} + \Sigma D_{D(x_i),y_i} + \Sigma D_{x_i,D(y_i)}$$

where by (2.8) nuclearity is preserved

$$z \in N(A) \Longrightarrow D(z) \in N(A)$$

$$3x + \Sigma[x_i,y_i] \in N(A) \Longrightarrow 3D(x) + \Sigma[D(x_i),y_i] + \Sigma[x_i,D(y_i)] \in N(A).$$

These show that Inder(A) and Stander(A) are ideals in Der(A).

2.19 Example. If A is associative all associators vanish, $A_{x,y} = 0$ and $D_{x,y} = D_{[x,y]}$, and all elements are nuclear. Thus Inder(A) = Stander(A) = Commder(A) consist of all commutator derivations D_z. ∎

2.20 Example. If $\frac{1}{3} \in \Phi$, and $N(A) = C(A)$ (as in a Cayley algebra) then by (2.18) Inder(A) = Stander(A) consists of all $\Sigma D_{x_i,y_i}$ since $D_z = 0$ for all central z. ∎

2.21 Example. If 3A = 0 then all D_x are derivations, $D_{x,y}$ reduces to $D_{[x,y]}$, and $D_x + \Sigma A_{x_i,y_i}$ is a derivation iff $\Sigma A_{x_i,y_i}$ is an associator derivation: Inder(A) = D_A + Assocder(A) $\supset$ Commder(A) = $D_A \supset D_{N(A)} + [A,A]$ = Stander(A). ∎

An inner derivation is called a **strictly inner derivation** if its indicator is strictly nuclear. The reason such derivations are "strict" is that they stay derivations in any extension $\tilde{A} \supset A$: they are derivations intrinsically, by their form alone. In contrast, an arbitrary inner derivation on A need not remain a derivation on $\tilde{A}$ since its indicator need not remain nuclear $\tilde{A}$.

When $D = D_z + \Sigma D_{x_i,y_i}$ has strictly nuclear indicator z we call it a **strictly standard** inner derivation. For example, all $D_{x,y}$ are strictly standard. If we denote by Inder(A), Strinder (A), Stander (A), and Strander (A) the spaces consisting of all inner, strictly inner, standard, and strictly standard derivations respectively we have inclusions

$$\mathrm{Der}(A) \supset \mathrm{Inder}(A) \begin{matrix} \supset \mathrm{Strinder}(A) \supset \\ \supset \mathrm{Stander}(A) \supset \end{matrix} \mathrm{Strander}(A)$$

Derivations into Bimodules

We have preferred to develope the theory of derivations acting on an algebra A rather than the more general theory of derivations of A into a bimodule M. We can reduce the general case of derivations into bimodules to the case of derivations of an algebra by means of the split null extension. A **derivation** of A into M is a linear map $A \xrightarrow{D} M$ satisfying the formal analogue of the condition for a derivation of an algebra,

$$D(ab) = D(a)\cdot b + a\cdot D(b) \qquad (a,b \in A).$$

Any derivation $A \xrightarrow{D} M$ extends to a derivation $E \xrightarrow{\tilde{D}} E$ of the split null extension $E = A \oplus M$ by $\tilde{D}(a \oplus m) = 0 \oplus D(a)$. Conversely, any derivation $\tilde{D}$ of E which kills M and maps A into M restricts to a derivation $A \xrightarrow{D} M$. Thus we can identify the space $Der(A,M)$ of derivations of A into M with a certain subspace of $Der(E)$.

An **inner derivation** of A into M is one which can be written in the form

$$D = D_n + \Sigma A_{x_i,n_i} \qquad (n,n_i \in M,\ x_i \in A) \tag{2.22}$$

where the **indicator** $z = 3n + \Sigma[x_i,n_i]$ belongs to the **nucleus** of the A-bimodule M. (Naturally enough, we define this nucleus to be $N(M) = \{n \in M \mid [n,A,A] = 0\}$; because M is a trivial ideal in E this is the same as the set of elements of M nuclear in E, $N(M) = M \cap N(E)$). D is **strictly inner** if its indicator is strictly nuclear (remains nuclear in all extensions $\tilde{E} \supset E$). For example,

$$D_{x,n} = D_{[x,n]} - 3A_{x,n} \tag{2.23}$$

is strictly inner with indicator zero. A **standard inner derivation** is one of the form

$$D = D_n + \Sigma D_{x_i,n_i} \qquad (n \in N(M),\ n_i \in M_i\ x_i \in A); \tag{2.24}$$

it is **strictly standard** if n is strictly nuclear. Once more <u>all inner derivations are standard when $\frac{1}{3}$ exists</u>,

$$D_n + \Sigma A_{x_i,n_i} = \frac{1}{3}\{D_m - \Sigma D_{x_i,n_i}\} \quad (m = 3n + \Sigma[x_i,n_i] \in N(M)) \tag{2.25}$$

In general we have the inclusions

$$Der(A,M) \supset Inder(A,M) \begin{matrix} \supset Strinder(A,M) \supset \\ \supset Strander(A,M) \supset \end{matrix} Strander(A,M).$$

Covering Derivations by Automorphisms

The standard automorphisms $T_{x,y}$ we introduced earlier are intimately connected with the standard inner derivations $D_{x,y}$, a fact which will be important for the proof of Malcev's Theorem. Indeed, the $D_{x,y}$'s are "infinitesimal generators" of the $T_{x,y}$'s.

2.26 (Infinitesimal Generation Lemma). Let x be an invertible element and y an element of the form $y = 1-z$ where z is nilpotent. Then

$$T_y \equiv I-D_z,\ T_{x,y} \equiv I-D_{x,zx^{-1}}$$

modulo multiplications involving two or more z's.

Proof. Always working modulo terms of degree ≥ 2 in z, and keeping in mind the distinction between the group commutator $[[xy]]$ and the algebra commutator $[x,z]$, we have

$$y^{-1} \equiv 1+z,\ xy \equiv x-xz,\ yx \equiv x-zx,$$

$$(xy)^{-1} \equiv x^{-1}+zx^{-1},\ (yx)^{-1} \equiv x^{-1}+x^{-1}z,\ [[xy]] \equiv 1-[x,zx^{-1}].$$

Indeed since $y^{-1} = (1-z)^{-1} = 1+z+z^2 \cdots + z^{k-1} \equiv 1+z$ if $z^k = 0$ (z is nil!) we get $(xy)^{-1} = y^{-1}x^{-1} \equiv (1+z)x^{-1}$ and $(yx)^{-1} = x^{-1}y^{-1}$ $\equiv x^{-1}(1+z)$, hence $[[xy]] = (xy)(yx)^{-1} \equiv (x-xz)(x^{-1}+x^{-1}z)$ $\equiv 1+z - xzx^{-1}$ (neglecting $xzx^{-1}z$) $= 1-x(zx^{-1}) + (zx^{-1})x = 1-[x,zx^{-1}]$.

By (1.3)

$$T_y = L_y R_{y^{-1}} \equiv (I-L_z)(I+R_z) \equiv I-L_z+R_z = I-D_z.$$

By (1.15)

$$T_{x,y} = L_{[[xy]]} R^{-1}_{[[xy]]} \{L_{(xy)^{-1}} L_x L_y\}^3.$$

Here

$$R_{[[xy]]} = R_{1-[x,zx^{-1}]} \equiv I - R_{[x,zx^{-1}]}$$

$$L_{[[xy]]} \equiv I - L_{[x,zx^{-1}]}$$

$$L_{xy}^{-1} L_x L_y \equiv \{L_x{-1} + L_{zx}{-1}\}L_x\{I-L_z\}$$

$$\equiv I + L_{zx}{-1} L_x - L_z \qquad \text{(neglecting } -L_{zx}{-1} L_x L_z\text{)}$$

$$= I - \{L_{(zx^{-1})x} - L_{zx}{-1} L_x\}$$

$$= I - A_{zx^{-1},x}$$

$$= I + A_{x,zx}{-1}.$$

In a product of terms $I+M_i$ for M_i a multiplication involving z_i we have $\Pi(I+M_i) \equiv I+\Sigma M_i$ modulo terms involving two or more z's. Therefore

$$T_{x,y} = L_{[[xy]]} R_{[[xy]]}{-1} \{L_{xy}^{-1} L_x L_y\}^3$$

$$= \{I - L_{[x,zx^{-1}]}\}\{I - R_{[x,zx^{-1}]}\}^{-1}\{I + A_{x,zx}{-1}\}^3$$

$$\equiv I - L_{[x,zx^{-1}]} + R_{[x,zx^{-1}]} + 3A_{x,zx}{-1}$$

$$= I - \{L_{[x,zx^{-1}]} - R_{[x,zx^{-1}]} - 3A_{x,zx}{-1}\}$$

$$= I - D_{x,zx}{-1}. \blacksquare$$

2.27 (Derivation Covering Theorem) Any standard inner derivation $D = D_z + \Sigma D_{x_i,z_i x_i^{-1}}$ for x_i invertible, z_i nil, and z nil in the nucleus of an alternative algebra is covered by a standard inner automorphism $T = T_y \Pi T_{x_i,y_i}$ for $y = 1-z$ invertible in the nucleus and $y_i = 1-z_i$ invertible:

$$T = T_y T_{x_1,y_1} \cdots T_{x_m,y_n} \equiv I - D_z - \Sigma D_{x_i,z_i x_i^{-1}} = I - D$$

modulo higher terms (multiplications involving two or more z or z_i's). $\blacksquare$

The reason for describing D as an "infinitesimal generator" of T if $T \equiv I+D$ is that the exponential

$$\exp(D) = \sum_{n=0}^{\infty} \frac{D^k}{k!}$$

of a derivation is always an automorphism (when it makes sense - for example, if D is nilpotent so the sum is actually finite, and the characteristic is zero so we can divide by k!). Thus saying $T \equiv I+D$ is saying $T \equiv \exp(D)$ mod higher terms, so T is "generated" in a suitable sense by D.

Exercises III.2

2.1 Verify (2.2)-2.5) in detail.

2.2 If L is a Lie algebra (a linear algebra where $x^2 = 0$ and $x(yz) + y(zx) + z(xy) = 0$ for all x,y,z) show the inner derivations are precisely all $L_x (= -R_x)$.

2.3 The D_z for $3z \in N(A)$ are called **commutator derivations**, forming a space $\mathrm{Commder}(A) \subset LM(A)$. Show this space of commutator derivations is an ideal in Der(A). Show the space of D_z with $3z = 0$ is also an ideal in Der(A). Does the set of D_z for z strictly nuclear form an ideal in Der(A)?

2.4 The map $D = \Sigma A_{x_i,y_i}$ is called an **associator derivation** if $\Sigma[x_i,y_i] \in N(A)$, and is **proper** if $\Sigma[x_i,y_i] = 0$. Show the space $\mathrm{Assocder}(A) \subset LM(A)$ of associator derivations forms an ideal in Der(A), as does the space of proper associator derivations. If $[x,y] \in N(A)$ then $A_{x,y}$ is an associator derivation. Does the collectio of all finite sums $\Sigma A_{x_i,y_i}$ where all $[x_i,y_i]$ are nuclear form an ideal in Der(A)?

2.5 Prove (I.3.7) and (I.3.8) using the commutator and associator derivation formulas twice on $U_x y = x(yx)$.

2.6 We noted Artin's Principle shows $[z,x^2] = x\circ[z,x]$; does it show D_z is a derivation?

2.7 Using a scalar extension argument and the fact that the coefficient of λ in $(x+\lambda y)^3$ is $U_x y + x^2 \circ y$, show that if D is a linear map satisfying $D(x^2) = x\circ Dx$, $D(x^3) = U_x D(x) + x^2 \circ D(x)$ in all extensions, then D is a derivation of the Jordan structure. If 2 is injective, use $2U_x y = x\circ(x\circ y) - x^2 \circ y$ to show $D(x^2) = x\circ D(x)$ is enough. Prove D_z and $A_{z,w}$ act as derivations on squares and cubes, so are derivations of the Jordan structure.

2.8 If $D = \Sigma A_{x_i,y_i} + \Sigma D_{z_j,w_j}$ where $\Sigma[x_i,y_i] = 0$, show D has indicator zero. Conversely, show an inner $D_x + \Sigma A_{x_i,y_i}$ with indicator zero has such form iff x is a sum of commutators $\Sigma[z_j,w_j]$. Show always 3x is a sum of commutators, so when $\frac{1}{3} \in \Phi$ all inner derivations with indicator zero have such a form.

2.9 Establish the alternate descriptions of the standard $D_{x,y}$:

$$D_{x,y} = [L_x,L_y] + [R_x,R_y] + [R_x,L_y] = [V_x,V_y] - [L_x,R_y]$$
$$= A^+_{x,y} - A_{x,y} \quad (\text{for } A^+_{x,y} = [L^+_x,R^+_y] = [V_x,V_y]).$$

2.10 An operator $D^+ = \Sigma[V_{x_i},V_{y_i}] = \Sigma A^+_{x_i,y_i}$ is called a **Jordan inner derivation**; show it is a derivation of the Jordan structure, and is an alternative derivation iff $\Sigma[x_i,y_i]$ is nuclear.

2.11 We can also describe the inner derivations in terms of the operators $V_{x,y}$ defined by $V_{x,y}(z) = U_{x,z}(y)$. Show

$$V_{x,y} = L_xL_y + R_{yx} = L_{xy} + R_xR_y = V_xV_y - U_{x,y} = L_{xy} + R_{yx} - A_{x,y}.$$

Find an expression for $\Sigma V_{x_i,y_i}$ and use it to show every element W of the Lie multiplication algebra has the form

$W = L_z + R_w + \Sigma V_{x_i,y_i}$. Show W is a derivation iff $z+w+x+y = 0$ (for $x = \Sigma x_iy_i$, $y = \Sigma y_ix_i$) and $x + 2z-w$ is nuclear. Conclude $\Sigma V_{x_i,y_i}$ is a derivation iff $\Sigma x_iy_i = -\Sigma y_ix_i \in N(A)$.

2.12 Show the Jordan associator $A^+_{x,y} = [V_x,V_y]$ may be expressed as $V_{x,y} - V_{y,x} = D_{x,y} + A_{x,y} = D_{[x,y]} - 2A_{x,y}$. Conclude anew that $A^+_{x,y}$ is a derivation of Jordan structure, and that $\Sigma V_{x_i,y_i}$ is a Jordan derivation iff $\Sigma x_i \circ y_i = 0$, in which case

$$2\Sigma V_{x_i,y_i} = \Sigma\{V_{x_i,y_i} - V_{y_i,x_i}\} = \Sigma A^+_{x_i,y_i}.$$

2.13 Define the **Lie multiplication algebra** LM(A,M) of A on a bimodule M to be the LM(A)-module of linear transformations of A into M generated by all left and right multiplications

L_m and R_m for $m \in M$ (the smallest subspace of $End_\Phi(A,M)$ containing these multiplications and closed under bracketing with all ℓ_a and r_a for $a \in A$ - note both A and M are A-bimodules). Show

$$LM(A,M) = L_M + R_M + [L_A, R_M]$$

$$Inder(A,M) = Der(A,M) \cap LM(A,M)$$

Show also $Inder(A,M) = Der(A,M) \cap \{LM(E)\}|_A$ for $E = A \oplus M$. These justify our definition of inner derivations into M.

2.14 Show $[[x,y]^2, xa, b] = [[x,y]^2, a, b]x$. How would you prove $[[x,y]^2, ax, b] = x[[x,y]^2, a, b]$? Show $[[x,y]^2, x, b] = 0$.

2.15 Compute $T_{x,y}$ modulo terms of degree 2 in z using the expression for $T_{x,y}$ in terms of the U's instead of the L's. (See Problem Set III.1.1 Ex.5).

III.2.1 Problem Set on Associator Maps

1. In addition to $A_{x,y} = L_{xy} - L_x L_y$ we introduce $B_{x,y} = L_{xy} - L_y L_x$. Show (i) $A_{x,x} = B_{x,x} = 0$, (ii) $A_{x,y} = -A_{y,x}$, $B_{x,y} = -B_{y,x}$, (iii) $A_{x,y} + B_{x,y} = L_{[x,y]}$, $B_{x,y} - A_{x,y} = [L_x, L_y]$, (iv) $L_x A_{x,y} = A_{x,yx} = A_{x,y} R_x$, $R_x A_{x,y} = A_{x,xy} = A_{x,y} L_x$, (v) $A_{x,y} B_{x,y} = B_{x,y} A_{x,y} = 0$, (vi) $(xy)[x,y,z] = y\{x[x,y,z]\}$, $[x,y,z](xy) = \{[x,y,z]y\}x$, $[x,y] \circ [x,y,z] = 0$.

2. Obtain analogous results for $C_{x,y} = R_{yx} - R_y R_x$ in place of $B_{x,y}$.

3. Show $R_x B_{x,y} = B_{x,y} R_x$, $L_x B_{x,y} = B_{x,xy}$, $B_{x,y} L_x = B_{x,yx}$.

4. Show (i) $A^2_{x,y} = L_{[xy]} A_{x,y} = A_{x,y} L_{[xy]} = -[L_x, L_y] A_{x,y} = -A_{x,y}[L_x, L_y]$ and (ii) $A^2_{x,y} = -R_{[xy]} A_{x,y} = -A_{x,y} R_{[xy]} = -[R_x, R_y] A_{x,y}$ $= -A_{x,y}[R_x, R_y]$.

5. Show that if $[x,y] = 0$ then $D = A_{x,y}$ is a derivation with $D^2 = 0$. Show $2D(z)^2 = 0$ for all z. Conclude $[x,y,z]^2 = 0$ if x commutes with A, and for such x also $[x^3,y,z] = 0$.

6. Use the operators $A_{x,y}$ to prove the left fundamental formula.

7. If $z = [x,y]$ show $V_z A_{x,y} = A_{x,y} V_z = 0$ ($V_a b = a \circ b$). Show $A_{z^2,x} = A_{z^2,y} = 0$. Linearize #1(v) and apply to z^2 to show $B_{x,y} A_{z^2,a} = 0$. Show $A_{x,y} A_{z^2,a} = 0$. Conclude $L_z A_{z^2,a} = 0$. Show $z[z^2,a,b] = [z^2,a,b]z = 0$.

8. Prove the <u>4th Power Theorem</u>: the 4th power of any commutator lies in the nucleus, $[x,y]^4 \in N(A)$.

III.2.2 Problem Set on the Bruck-Kleinfeld f-Function

The f function on an alternative algebra is the 4-linear mapping

$$f(x,y,z,w) = [xy,z,w]-[y,z,w]x-y[x,z,w].$$

This almost measures how far $A_{z,w}$ is from being a derivation (if xy were replaced by yx in the first associator); instead, it measures how far $A_{z,w}$ is from being an anti-derivation $D(xy) = D(y)x+yD(x)$. The standard method of proving identities in alternative algebras is by using the properties of the f-function. We indicate this in the following problems.

We could have mentioned this early in Chapter I, but we have preferred to keep the development as free from formulas as possible. Thus we have swept the f-function under the rug until now.

1. Show that in an alternative algebra f is an alternating function of its arguments. (Use only alternativity, and the associator formula II.2.7, not our basic identities).
2. Deduce the bumping formulas and then Moufang's identities.
3. Show $F(x,y,z,w) = [x,[yz,w]]-[y,[z,w,x]]+[z,[w,x,y]]-[w,[x,y,z]]$ equals $f(x,y,z,w)-f(y,z,w,x)+f(z,w,x,y)$. Using exercise 1 deduce also $F = 3f$. Alternately, use the first equality to prove exercise 1.
4. From exercise 1 show $f(x,y,z,w) = [[x,y],z,w]+[[z,w],x,y]$.
5. Deduce the Associator Derivation Formula.
6. Show $f(zw,z,x,y) = zf(w,z,x,y)+[z,x,y][w,z]$.
7. Another way to show f is alternating would be to show $f(x,y,z,w) = [xy,z,w]-[yz,w,x]+[zw,x,y]-[wx,y,z]$ is just the alternating sum generated by $[xy,z,w]$.

III.2.3 Problem Set on the Structure Algebra

Just as the structure group corresponds to autotopies, and is obtained by tacking on multiplications to the automorphism group, so the structure algebra corresponds to diffeotopies, and is obtained by adding on multiplications to the derivation algebra.

1. A **diffeotopy** or **local autotopy** of a unital alternative algebra A is a triple $(W,W',W")$ of linear transformations (not necessarily invertible) such that

   $$(*) \qquad W(xy) = W'(x)y+xW"(y) \qquad (x,y \in A).$$

   This is a generalization of the derivation condition (2.1), so W is a sort of generalized derivation. Just as automorphisms were a special kind of autotopy, show derivations are a special kind of diffeotopy:

   D is a derivation iff (D,D,D) is a diffeotopy.

2. The diffeotopies form a Lie algebra just as the derivations do: show the diffeotopies form a Lie subalgebra of $\mathrm{End}(A)^- \times \mathrm{End}(A)^- \times \mathrm{End}(A)^-$.

3. Besides derivations show we also have diffeotopies determined by multiplications:

   $(L_z,V_z,-L_z)$, $(R_z,-R_z,V_z)$, (V_z,L_z,R_z) are diffeotopies.

4. Show the three entries in a diffeotopy are related by

   $$(W,W',W") = (W,W-R_{w"},W-L_{w'}) \quad (w' = W'(1), w" = W"(1))$$

   (Notice that as a consequence if $W'1 = W"1 = 0$ then W is a derivation). From this show we can permute a diffeotopy, if $(W,W',W")$ is a diffeotopy so are $(W',W,W"-V_{w"})$ and $(W",W'-V_{w'},W)$ $\quad (w' = W'(1), w" = W"(1))$.

5. The <u>structure algebra</u> Strl(A) consists of all linear transformations W on A for which there exist W',W" as in (*), i.e. all W which appear as the first entry of a diffeotopy. (Show it doesn't matter which entry W appears in). In operator notation, find the defining conditions for W to appear as a second or third entry (i.e. the defining conditions for W' or W" in (*)).

6. Show Strl(A) is a Lie algebra of linear transformations on A. Show this Lie algebra contains all derivations D, as well as all multiplications L_x and R_x, hence the whole Lie multiplication algebra (generated by the L_x and R_x)

$$\mathrm{Strl}\ (A) \supset LM(A) + \mathrm{Der}\ (A).$$

7. In general Strl(A) doesn't contain much more than this; establish the <u>Proposition</u>. If $\frac{1}{3} \in \Phi$ then every W in the structure algebra has the form

$$W = L_x + R_y + D$$

for some $x,y \in A$ and some derivation D. Thus

$$\mathrm{Strl}\ (A) = L_A + R_A + \mathrm{Der}\ (A)$$

8. The **inner structure algebra** is

$$\mathrm{Instrl}\ (A) = \mathrm{Strl}\ (A) \cap LM(A) = LM(A).$$

If $1/3 \in \Phi$ deduce

$$\mathrm{Instrl}\ (A) = L_A + R_A + \mathrm{Inder}\ (A) \qquad (1/3 \in \Phi).$$

9. Show that although the first entry does not determine exactly the rest of the diffeotopy, it does up to translations from the nucleus; if (W,W',W") is a diffeotopy then $(W,\tilde{W}',\tilde{W}")$ is another diffeotopy iff $\tilde{W}' = W'+R_n$, $\tilde{W}" = W"-L_n$ for $n \in N(A)$. Modulo this uncertainty, show $W \to W'$ and $W \to W"$ are automorphisms of

the structure algebra. Make this precise by forming Strl (A)/N for N the ideal generated by all L_n and R_n for $n \in N(A)$.

10. Show $W \in$ Strl (A) is a derivation iff $W'(1) = n$, $W''(1) = -n$ for nuclear n.

11. Prove directly from the definitions that if (W, W', W'') is a diffeotopy so is $(W', W'+R_{W''}, W''-V_{W''})$.

12. Verify $D_{x,y} = L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y]$ is a derivation by showing $(D_{x,y}, D_{x,y}, D_{x,y})$ is a diffeotopy.

13. Do the U_x belong to the structure algebra?

14. Show that $W \in$ Strl (A) is a derivation of Jordan structure iff $W1 = 0$.

III.2.4 Problem Set: Alternate Proofs using Dual Numbers

Let $\Omega = \Phi[\varepsilon]$ $(\varepsilon^2 = 0)$ be the ring of dual numbers over Φ. We will derive the properties of diffeotopies from those autotopies, established in Problem Set III.1.2 (hereafter denoted SG).

1. If W is any linear transformation on A, show that $I+\varepsilon W$ is invertible on A_Ω with inverse $I-\varepsilon W$. Show that (W,W',W'') is a diffeotopy of A iff $(I+\varepsilon W, I+\varepsilon W', I+\varepsilon W'')$ is an autotopy of A.
2. Use the "standard" trick with $\Psi = \Phi[\varepsilon_1,\varepsilon_2]$ for $\varepsilon_1^2 = \varepsilon_2^2 = 0$, $\varepsilon_1\varepsilon_2 = \varepsilon$ to show that the closure of autotopies of A under products implies closure of diffeotopies of A under Lie brackets: (W,W',W''), (V,V',V'') diffeotopies implies $([WV],[W'V'],[W''V''])$ is too.
3. Use SG Ex. 3 to show that if (W,W',W'') is a diffeotopy then $(W,W',W'') = (W,W-R_s,W-L_r)$ for $r = W'1$, $s = W''1$.
4. Use SG Ex. 8 to show that if (W,W',W'') and $(W,\tilde{W}',\tilde{W}'')$ are diffeotopies then $\tilde{W}' = W'+R_n$, $\tilde{W}'' = W''-L_n$ for $n \in N(A)$.
5. Use SG Ex. 2 to deduce that $(L_x,V_x,-L_x)$, $(R_x,-R_x,V_x)$, (V_x,L_x,R_x) are diffeotopies for any x. Conclude as before that if (W,W',W'') is a diffeotopy so is $(W',W,W''+V_{\bar{s}})$ and $(W'',W'-V_r,W)$ for $r = W'1$, $s = W''1$.
6. Show that D is a derivation of A iff $I+\varepsilon D$ is an automorphism of A_Ω; conclude from SG Ex. 1 that D is a derivation iff (D,D,D) is a diffeotopy.

7. Use SG Ex. 7 to show that if $1/3 \in \Phi$ and $W \in Str(A)$ then $W = L_x + R_y + D$ for D a derivation.

In the next few exercises we indicate how certain derivations D can be obtained from automorphism $T = I + \varepsilon D$.

8. Establish the following expressions for the inner derivation $D_{x,y}$:

$$\begin{aligned} D_{x,y} &= L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y] \\ &= [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \\ &= [V_x, V_y] - [L_x, R_y]. \end{aligned}$$

9. Show that, in any group, cd commutes with dc iff $[[c^{-1}d^{-1}]] = [[cd]]$ (group commutator). Show that if $[[cd, dc]] = 1$ for invertible $c, d \in A$ then $T = [[U_c, U_d]] \cdot [[L_c, R_d^{-1}]]$ is an automorphism of A.

10. Over $\Psi = \Phi[\varepsilon_1, \varepsilon_2]$ $(\varepsilon_1^2 = \varepsilon_2^2 = 0,\ \varepsilon_1\varepsilon_2 = \varepsilon)$ the elements $c = 1 + \varepsilon_1 x$, $d = 1 + \varepsilon_2 y$ have $[[c, d]] = [[c^{-1}, d^{-1}]]$ and $T = [[U_c, U_d]][[L_c, R_d^{-1}]] = I + \varepsilon D_{x,y}$.

11. Use arguments similar to Ex. 9 to show that if c and d commute then $T = [[U_c, U_d]]$, $T = [[L_c, R_d]]$, $T = [[L_c, U_d]]$ are automorphisms; that if $c = 1 + \varepsilon_1 x$, $d = 1 + \varepsilon_2 y$ where x, y commute then c, d commute; hence that if x, y commute then $[V_x, V_y]$, $[L_x, R_y]$, $[L_x, L_y]$ (and therefore also $[R_x, R_y]$) are derivations.