§3. Automorphisms and derivations of $\mathbb{C}(B,\mu)$

The automorphisms of a Cayley algebra $\mathbb{C}(B,\mu)$ which fix the quaternion algebra B are obtained by shifting by an element of norm 1, and all such automorphisms are inner. Analogously, the derivations of $\mathbb{C}(B,\mu)$ killing B are obtained by shifting by an element of trace zero, and all such derivations are inner.

Automorphisms

We begin by characterizing the subgroup $Aut_B(\mathbb{C})$ of automorphisms of a Cayley algebra $\mathbb{C}(B,\mu)$ over a field Φ which fix the quaternion algebra B.

3.1 (Quaternion Dilation Theorem) The automorphisms of a Cayley algebra $\mathbb{C}(B,\mu)$ which fix the quaternion subalgebra B elementwise are precisely the quaternion dilations

$$T_{B,u}: \quad a+b\ell \to a+(ub)\ell \qquad (n(u)=1)$$

by elements u of norm 1. Thus $Aut_B(\mathbb{C})$ is isomorphic to the multiplicative group of elements of norm 1 in B.

Proof. All maps $T = T_{B,u}$ are automorphisms by direct calculation: from the multiplication formula I.1.8 $T(x_1x_2)$ $= T(\{a_1a_2 + \mu\bar{b}_2b_1\} + \{b_2a_1 + b_1\bar{a}_2\}\ell) = \{a_1a_2 + \mu\bar{b}_2b_1\}$ $+ \{u(b_2a_1 + b_1\bar{a}_2)\}\ell = \{a_1a_2 + \mu(\overline{ub_2})(ub_1)\} + \{(ub_2)a_1 + (ub_1)\bar{a}_2\}\ell$ (since $\bar{u}u = n(u)1 = 1$ and B is associative) $= \{a_1 + (ub_1)\ell\}\cdot$ $\{a_2 + (ub_2)\ell\} = T(x_1)T(x_2)$, and T is clearly a bijective linear map since $b \to ub$ is bijective on B. In fact, $T^{-1}_{B,u} = T_{B,u^{-1}}$ since in general $T_{B,u_1}T_{B,u_2} = T_{B,u_1u_2}$ and $T_{B,1} = I$. Thus the quaternion dilations $T_{B,u}$ form a group isomorphic to the multiplicative group of quaternions u of norm 1.

These dilations are precisely all automorphisms which fix B. Any T which fixes B must have the form $T(x) = T(a+b\ell) = T(a) + T(b)T(\ell) = a + bT(\ell)$ ($a, b \in B$ are fixed). We need only show $T(\ell) = u\ell$ for some u with $n(u) = 1$, since then $b(u\ell) = (ub)\ell$ and $T = T_{B,u}$ as claimed. If we write $T(\ell) = v + u\ell$ then $va + (u\bar{a})\ell = T(\ell)a = T(\ell a) = T(\bar{a}\ell) = \bar{a}T(\ell) = \bar{a}v + (u\bar{a})\ell$ implies $va = \bar{a}v$ for all $a \in B$, therefore $[a,b]v = (ab)v - b(av) = v(\overline{ab}) - bv\bar{a} = v\bar{b}\bar{a} - v\bar{b}\bar{a} = 0$ and consequently $v = 0$ since there are invertible commutators $[a,b]$ in a quaternion algebra B. (B is either a noncommutative division ring or $M_2(\Phi)$; in the first case any $[x,y] \neq 0$ is invertible, in the second $[e_{12},e_{21}] = e_{11} - e_{22}$ is invertible). Then $T(\ell) = u\ell$ where $\mu 1 = T(\mu 1) = T(\ell^2) = T(\ell)^2 = (u\ell)^2 = \mu\bar{u}u = \mu n(u)1$ implies $n(u) = 1$. (Alternately, one can show T is an isometry relative to the norm form n, hence $T(\ell) \in T(B^{\perp}) \subset B^{\perp} = B\ell$, so $T(\ell) = u\ell$ where $n(\ell) = n(T\ell) = n(u\ell) = n(u)n(\ell)$ forces $n(u) = 1$). ∎

Next we show these quaternion dilations are inner.

3.2 Proposition. If u is a product of commutators $[[xy]]$ in B then the quaternion dilation $T_{B,u}$ is inner.

Proof. Since $T_{B,u} = T_{B,u_1} \cdots T_{B,u_n}$ if $u = u_1 \cdots u_n$ for $u_i = [[x_i y_i]]$, where automatically $n([[xy]]) = n(x)n(y)n(x)^{-1}n(y)^{-1} = 1$, it suffices if each T_{B,u_i} is inner. Thus we may assume $u = [[x,y]]$. We claim

$$T_{B,[[x,y]]} = L_{(xy)\ell,\ell}\, L_{x^{-1}\ell,\, y^{-1}\ell}$$

From our general formula (1.4) for the action of $L_{a\ell,\overline{b}\ell}$ we see

$L_{(xy)\ell,\ell}L_{x^{-1}\ell,\overline{y}^{-1}\ell}(a+b\ell) = L_{(xy)\ell,\ell}\{(y^{-1}x^{-1})^{-1}a(y^{-1}x^{-1}) + (x^{-1}y^{-1})b(y^{-1}x^{-1})^{-1}\ell\} = L_{(xy)\ell,\ell}\{(xy)a(xy)^{-1} + (yx)^{-1}b(xy)\cdot\ell\}$
$= (xy)^{-1}\{(xy)a(xy)^{-1}\}(xy) + (xy)\{(yx)^{-1}b(xy)\}(xy)^{-1}\cdot\ell$
$= a + \{(xy)(yx)^{-1}b\}\ell = a + \{[[xy]]b\}\ell = T_{B,[[xy]]}(a + b\ell)$ as claimed. ∎

It is known to group theorists that the derived group of the quaternions of norm $\neq 0$ is the group of quaternions of norm $= 1$ (for the split quaternion algebra $M_2(\Phi)$ this means the derived group ofthe general linear group $GL(2,\Phi)$ of matrices of determinant $\neq 0$ is the special linear group $SL(2,\Phi)$ of matrices of determinant 1), except when $\Phi = \mathbb{Z}_2$. We will give a direct proof that when $\Phi \neq \mathbb{Z}_2$ every quaternion of norm 1 actually is a commutator (and not merely a product of commutators).

3.3 (Norm 1 Quaternion Lemma) Every element of norm 1 in a quaternion algebra B is a commutator

$$u = [[xy]]$$

except when $B \cong M_2(\mathbb{Z}_2)$, $u \cong \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Proof. Any central $u = \alpha 1 \varepsilon \Phi 1$ is easily seen to be a commutator: $n(u) = \alpha^2 = 1$ implies $\alpha = \pm 1$, $u = \pm 1$, where 1 is trivially a commutator, hence so is -1 in characteristic 2, whereas in characteristic $\neq 2$ the quaternion algebra has basis $1,i,j,k$ with $ij = -ji$ so $[[ij]] = (ij)(ji)^{-1} = -(ji)(ji)^{-1} = -1$

Another case where we can explicitly represent u as a commutator is when $t(u) = -2$, with the exception indicated in the proposition. If $u = -1$ this follows from the previous paragraph, so assume $t(u) = -2$, $n(u) = 1$, $u \neq -1$. If we set $z = u+1$ then $t(z) = t(u) + 2 = 0$, $n(z) = 1+t(u) + 1 = 0$, yet $z \neq 0$. Thus the quaternion algebra B is not a division algebra, consequently $B \cong M_2(\Phi)$ is split. Since u has minimum polynomial $\lambda^2 - t(u)\lambda + n(u) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$ with characteristic roots in Φ, by Jordan canonical form we can choose the isomorphism $B \cong M_2(\Phi)$ so $u \cong \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. But this matrix is a commutator in $M_2(\Phi)$ for any field $\Phi \neq \mathbb{Z}_2$: $u = a^{-1}b^{-1}ab$ for $a = \begin{pmatrix} 1 & 0 \\ 2\lambda & \lambda-1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & -\lambda^{-1} \\ 0 & -1 \end{pmatrix}$ for any $\lambda \in \Phi$ with $\lambda \neq 0,1$ since

$$ab = \begin{pmatrix} 1 & 0 \\ 2\lambda & \lambda-1 \end{pmatrix}\begin{pmatrix} 1 & -\lambda^{-1} \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\lambda^{-1} \\ 2\lambda & -2-\lambda+1 \end{pmatrix}$$

$$bau = \begin{pmatrix} 1 & -\lambda^{-1} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2\lambda & \lambda-1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\lambda^{-1} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -2\lambda & \lambda+1 \end{pmatrix} = \begin{pmatrix} -1+2 & 1-(\lambda+1)\lambda^{-1} \\ 2\lambda & -\lambda-1 \end{pmatrix}$$

where $1-(\lambda+1)\lambda^{-1} = \{\lambda-(\lambda+1)\}\lambda^{-1} = -\lambda^{-1}$. Therefore u is a commutator unless $B \cong M_2(\Phi)$, $u \cong \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, $\Phi = \{0,1\} = \mathbb{Z}_2$.

From now on assume $u \notin \Phi 1$ and $t(u) \neq -2$. From $t(u) \neq -2$ we see $x = 1+u$ is invertible: $n(x) = n(1+u) = 1 + t(u) + n(u) = 2 + t(u) \neq 0$. Furthermore, from $x \notin \Phi 1$ we can find an invertible commutator $y = [u,v]$, because if the quadratic function $f(x) = n([u,x])$ vanished identically on B it would vanish on the split quaternion algebra $B_\Omega \cong M_2(\Omega)$ (Ω the algebraic closure of Φ; see I.2.4), and in the split case if $\det[u,e_{12}] = \det[u,e_{21}] = \det[u,e_{12}+e_{21}] = 0$ then

$u = \Sigma_{i,j=1}^{2} \alpha_{ij} e_{ij}$ is a scalar: if $0 = \det[u, e_{12}]$
$= \det(\alpha_{11}e_{11}e_{12} + \alpha_{21}e_{21}e_{12} - \alpha_{21}e_{12}e_{21} - \alpha_{22}e_{12}e_{22})$
$= \det\begin{pmatrix} -\alpha_{21} & \alpha_{11}-\alpha_{22} \\ 0 & \alpha_{21} \end{pmatrix} = -\alpha_{21}^2$ then $\alpha_{21} = 0$, similarly $\alpha_{12} = 0$ if
$0 = \det[u, e_{21}]$, and then if $0 = \det[u, e_{12}+e_{21}]$
$= \det[\alpha_{11}e_{11} + \alpha_{22}e_{22}, e_{12} + e_{21}] = \det\begin{pmatrix} 0 & \alpha_{11}-\alpha_{22} \\ \alpha_{22}-\alpha_{11} & 0 \end{pmatrix}$
$= -(\alpha_{11}-\alpha_{22})^2$ we would have $\alpha_{11} = \alpha_{22} = \alpha$ and $u = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \alpha 1$.

Anytime $n(u) = 1$ we have $u[u,v](1+u) = (1+u)[u,v]$ for all v, since $-u[\overline{u},v](1+u) = -u\overline{u}v(1+u) + uv\overline{u}(1+u) = -u\overline{u}v - vu + uv\overline{u} + uv$ (using $u\overline{u} = \overline{u}u = 1$) $= [u,v] - u[\overline{u},v] = (1+u)[u,v]$. Thus u is a commutator $(1+u)[u,v](1+u)^{-1}[u,v]^{-1} = [[xy]]$ long as $x = 1+u$ is invertible and some $y = [u,v]$ is invertible. By our previous results this shows any $u \notin \Phi 1$ with $t(u) \neq -2$ is a commutator, and all u with $n(u) = 1$ are commutators except for the case $u \cong \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $B \cong M_2(\mathbb{Z}_2)$. ∎

Even though in the exceptional case u is not a commutator, the dilation $T_{B,u}$ is still inner. Indeed, in $B = M_2(\mathbb{Z}_2)$ the element $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ can be written as $1+\mu xy$ where $yx = 0$ for $x = \mu^{-1}e_{11}$, $y = e_{12}$ Therefore $T_{B,u}$ is inner by the general

3.4 Proposition. If $yx = 0$ for $x, y \in B$ then the quaternion dilation by $u = 1+\mu xy$ is inner in $\mathbb{C}(B,\mu)$,

$$T_{B,1+\mu xy} = L_{1+x\ell,\ 1+\overline{y}\ell}.$$

Proof. Here $(1+x\ell)(1+\overline{y}\ell) = 1 + x\ell + \overline{y}\ell + \mu yx = 1+(x+\overline{y})\ell$ if $yx = 0$. One can compute directly that $L_{1+(x+\overline{y})\ell}\, T_{B,u}$ $= L_{1+x\ell} L_{1+\overline{y}\ell}$ on $a + b\ell$, hence $T_{B,u} = L_{1+(x+\overline{y})\ell}^{-1}\, L_{1+\overline{y}\ell}\, L_{1+\overline{y}\ell}$

$= L_{1+x\ell,\ 1+\bar{y}\ell}$. Alternately, $L_{1+x\ell}\, L_{1+\bar{y}\ell} = \{I + L_{x\ell}\}\{I + L_{\bar{y}\ell}\}$ $= I + L_{x\ell} + L_{\bar{y}\ell} + L_{x\ell}L_{\bar{y}\ell}$ where $L_{x\ell}\, L_{\bar{y}\ell}\,(a + b\ell) = L_{x\ell}(\,(\bar{y}a)\,\ell + \mu\bar{b}\bar{y})$ $= \mu a y x + \mu(xyb)\ell = \mu(xyb)\ell$ (since $yx = 0$) and therefore $L_{(x+\bar{y})\ell}\, L_{x\ell}\, L_{\bar{y}\ell}\,(a + b\ell) = L_{(x+\bar{y})\ell}(\mu xyb)\ell = \mu^2\, \bar{b}\bar{y}\bar{x}(x+\bar{y}) = 0$ (since $\bar{x}\bar{y} = \overline{yx} = 0$, and $yx = 0$ implies either $y = 0$ and $\bar{y}n(x) = 0$ or $y \neq 0$ and x is not invertible and $\bar{x}x = n(x)1 = 0$) so we can write $L_{1+x\ell}\, L_{1+\bar{y}\ell} = I + L_{(x+\bar{y})\ell} + \{I + L_{(x+\bar{y})\ell}\}L_{x\ell}L_{\bar{y}\ell}$ $= \{I + L_{(x+\bar{y})\ell}\}\{I + L_{x\ell}L_{\bar{y}\ell}\}$. Thus $L_{1+x\ell,1+y\ell}$ $= (I + L_{(x+\bar{y})\ell})^{-1}\, L_{1+x\ell}L_{1+\bar{y}\ell} = I + L_{x\ell}L_{\bar{y}\ell}$ sends $a + b\ell$ to $a + b\ell + \mu(xyb)\ell = a + (ub)\ell$ and coincides with $T_{B,u}$. ■

Since $T_{B,u}$ is inner for all commutators u by 3.2, hence for all u except $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ by 3.3, and since $T_{B,u}$ is inner for $u = \begin{pmatrix}1&1\\0&1\end{pmatrix}$ by 3.4, we see the quaternion dilations are always inner.

3.5 (Inner Dilation Theorem) All quaternion dilations $T_{b,u}$ are inner automorphisms of the Cayley algebra $\mathbb{C}(B,\mu)$ (indeed, all are products of associations). ■

3.6 Remark. In $B = M_2(\mathbb{Z}_2)$ the element $u = \begin{pmatrix}1&1\\0&1\end{pmatrix}$ is neither a commutator nor a product of commutators $[[xy]]$. In this B there are 16 elements, only 6 of which are invertible: $1 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $u = \begin{pmatrix}1&1\\0&1\end{pmatrix}$, $u' = \begin{pmatrix}1&0\\1&1\end{pmatrix}$, $u'' = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $v = \begin{pmatrix}1&1\\1&0\end{pmatrix}$, $v' = \begin{pmatrix}0&1\\1&1\end{pmatrix}$. One easily checks $u^2 = v^3 = 1$, $v' = v^2$, $u' = vu$, $u'' = uv = v^2u$ so the multiplicative group $G = \dot{B} = \{1, v, v^2, u, vu, uv\}$ is the semidirect product $G = K \times H$ of the normal subgroup $K = \{1, v, v^2\} \cong \mathbb{Z}(3)$ and the subgroup $H = \{1, u\} \cong \mathbb{Z}(2)$, where the automorphism $k \to uku^{-1}$ of K induced by H is $uvu^{-1} = v^2$. In this case $G = \{x \mid n(x) \neq 0 \text{ in } \mathbb{Z}_2\} = \{x \mid n(x) = 1 \text{ in } \mathbb{Z}_2\}$, but the commutator subgroup is a proper subgroup. Indeed, $[[G,G]]$ is generated by

$[[u,v]] = uvu^{-1}v^{-1} = v^2v^{-1} = v$ and its conjugate v^2 (in general, if g_i generate G then the conjugates $[[g_i,g_j]]^g$ generate $[[G,G]]$), so $[[G,G]] = \{1,v,v^2\} = K$. ■

3.7 Example. In characteristic $\neq 2$ the most important automorphisms are the quaternion reflections $T_B = T_{B,-1}$ (being reflection $a + b\ell \to a - b\ell$ in the quaternion subalgebra B). Such T_B are actually associations,

$$T_B = L_{i,j} = L_{ij}^{-1} L_i L_j$$

since if 1, i, j, k is a standard quaternion basis for B we have $L_{i,j}(a + b\ell) = (ij)^{-1}ij\ a + \{bji(ij)^{-1}\}\ell = a + \{b(-ij)(ij)^{-1}\}\ell$ $= a - b\ell = T_B(a + b\ell)$. ■

Derivations

Now we turn to the Lie subslgebra $\mathrm{Der}_B(\mathbb{C})$ of derivations of a Cayley algebra $\mathbb{C}(B,\mu)$ which annihilate B.

3.8 (Quaternion Translation Theorem) The derivations of $\mathbb{C}(B,\mu)$ which annihilate B are precisely the quaternion translations

$$D_{B,u}: \quad a + b\ell \longrightarrow (ub)\ell \qquad (t(u) = 0)$$

by elements u of trace zero. Thus $\mathrm{Der}_B(\mathbb{C})$ is isomorphic to the Lie algebra of elements of trace zero in B.

Proof. By direct calculation the quaternion translations $D = D_{B,u}$ are derivations: $D(x_1x_2) = D(\{a_1a_2 + \mu\bar{b}_2b_1\} + \{b_2a_1 + b_1\bar{a}_2\}\ell)$ $= u\{b_2a_1 + b_1\bar{a}_2\}\cdot\ell = \{ub_2a_1 + ub_1\bar{a}_2\}\ell$ coincides with $D(x_1)x_2 + x_1D(x_2) = \{(ub_1)\ell\}\{a_2 + b_2\ell\} + \{a_1 + b_1\ell\}\{(ub_2)\ell\}$ $= \mu\{\bar{b}_2ub_1 + \bar{b}_2\bar{u}b_1\} + \{ub_1\bar{a}_2 + ub_2a_1\}\ell$ iff $u + \bar{u} = t(u) = 0$.

Clearly $D_{B,u_1}D_{B,u_2}(a + b\ell) = (u_1u_2b)\ell$ so $[D_{B,u_1},D_{B,u_2}]$ $= D_{B,[u_1,u_2]}$ and the quaternion translations form a Lie algebra isomorphic to the trace zero elements of B.

These translations are precisely all the derivations which kill B. Indeed, and derivation D killing B has the form $D(a + b\ell) = D(a) + D(b)\ell + bD(\ell) = bD(\ell)$. We need only show $D(\ell) = u\ell$ for $t(u) = 0$, since then $D(a + b\ell) = b(u\ell) = (ub)\ell$ $= D_{B,u}(a + b\ell)$. If we write $D(\ell) = v + u\ell$ then $av + (ua)\ell = aD(\ell) = D(a\ell) = D(\ell\bar{a}) = D(\ell)\bar{a} = v\bar{a} + (ua)\ell$, and once more $av = v\bar{a}$ for all $a \in B$ forces $v = 0$. Thus $D(\ell) = u\ell$, where $0 = D(\mu 1) = D(\ell^2) = \ell \circ D(\ell) = \ell \circ u\ell = \mu(\bar{u}+u) = \mu t(u)1$ forces $t(u) = 0$. (Alternately, one can show D is alternating relative to the norm form, $n(Dx,x) = 0$, so $D(\ell) \in D(B^{\perp}) \subset B^{\perp} = B\ell$ by $n(D\ell,B) = -n(\ell,D(B)) = 0$, and if $D(\ell) = u\ell$ then $0 = n(D(\ell),\ell) = n(u\ell,\ell) = t(u)n(\ell)$ implies $t(u) = 0$). ∎

In showing these $D_{B,u}$ are inner we will need to express u in terms of (algebra) commutators $[x,y]$. It is well known to Lie algebraists that all trace zero quaternions are sums of commutators (for the split quaternion algebra $B \cong M_2(\Phi)$ this means the derived algebra of the Lie algebra $g\ell(2,\Phi) = M_2(\Phi)^-$ is the algebra $s\ell(2,\Phi)$ of trace zero elements). This is easy to see in the split case, since in $B = M_2(\Phi)$ Ker t is a 3-dimensional subspace containing all commutators (since $t(xy) = t(yx)$ implies $t([x,y]) = 0$), yet the commutators $e_{12} = [e_{11},e_{12}]$, $e_{21} = [e_{22},e_{21}]$, $e_{11} - e_{22} = [e_{12},e_{21}]$ already span a 3-dimensional subspace, so all trace zero elements are sums of commutators. In the unsplit case Ker t is still 3-dimensional, as is the derived algebra (over the algebraic closure Ω $[B,B]_\Omega = [B_\Omega, B_\Omega] = \text{Ker } t_\Omega$ is 3-dimensional over Ω, hence $[B,B]$ is 3-dimensional over Φ), so again Ker $t = [B,B]$.

Actually we have a stronger

3.9 (Trace Zero Quaternion Lemma) Every quaternion u of trace zero is a commutator, $u = [x,y]$.

Proof. Whenever $a, b \in B$ with $t(b) = 0$ we have
$[a\bar{b}, b] = an(b) - ba\bar{b} = n(b)a - b\{n(a,b) - b\bar{a}\}$
$= n(b)a - n(a,b)b + \{t(b)b - n(b)1\}\bar{a} = n(b)\{a-\bar{a}\} - n(a,b)b$, or

(3.10) $$[a\bar{b},b] = 2n(b)a - n(b)t(a)1 - n(a,b)b \qquad (t(b) = 0).$$

First consider the characteristic 2 case, where the first of the three terms drops out. If u is independent of 1 we can take $b = u$ (remember $t(u) = 0$!) and find $a \in B$ with $n(a,1) = 0$, $n(a,u) = -1$ (because the bilinear norm form $n(x,y)$ is nondegenerate on a quaternion algebra, and therefore the linear functionals $n(a,\cdot)$ separate independent vectors); then in (3.10) $2 = t(a) = 0$ and $n(a,u) = -1$ imply $[a\bar{u},u] = u$ is a commutator. On the other hand, when $u = \alpha 1$ is dependent on 1 we can find $b \in B$ independent of 1 with $t(b) = 0$, $n(b) \neq 0$ (there exists z independent of 1 in Ker t; if $n(z) \neq 0$ then $b = z$ will do, while if $n(z) = 0$ then $b = 1 + z$ has $t(b) = 2 + t(z) = 0$, $n(b) = 1 + t(z) + n(z) = 1$) and then by nondegeneracy again find a with $n(a,b) = 0$, $n(a,1) = -\alpha n(b)^{-1}$, so that by (3.10) $[a\bar{b},b] = 0 + n(b)\alpha n(b)^{-1} 1 - 0 = \alpha 1 = u$ is a commutator in this case as well.

Now assume the characteristic $\neq 2$. If $t(u) = n(u) = 0$ we can by nondegeneracy find $a \in B$ with $n(a,u) = -1$; setting $b = u$ in (3.10) we see $[a\bar{u},u] = 0 - 0 + u = u$ is a commutator. If $n(u) \neq 0$ then $U = \Phi 1 + \Phi u$ is a nondegenerate subspace of B under $n(x,y)$, so $B = U \oplus U^{\perp}$ where $U^{\perp}$ must also be nondegenerate if $n(x,y)$ is to be nondegenerate on all of B. In particular, we can find $b \in U^{\perp}$ with $n(b) \neq 0$. If we set $a = \{2n(b)\}^{-1}u$ (remember characteristic $\neq 2$!) then $t(u) = n(b,1) = n(b,u) = 0$ imply $t(a) = t(b) = n(b,a) = 0$ and in (3.10) $[a\bar{b},b] = u - 0 - 0 = u$. Thus in all cases a traceless element u is a commutator $u = [a\bar{b},b]$. ■

We can now show all quaternion translations are inner.

3.11 (Inner Translation Theorem) Every quaternion translation $D_{B,u}$ of a Cayley algebra $\mathbb{C}(B,\mu)$ is strictly inner with indicator zero:

$$D_{B,[x,y]} = \frac{1}{\mu}\{A_{y\ell,\bar{x}\ell} - A_{(xy)\ell,\ell}\}$$

Proof. By the previous lemma we can write the trace zero element u as $u = [x,y]$ for some $x,y \in B$. Now from our formula (2.14) we have $A_{c\ell,\bar{d}\ell}(a + b\ell) = \mu[dc,a] + \mu\{bdc - cdb\}\ell$ so

$$\begin{aligned}\frac{1}{\mu}\{A_{y\ell,\bar{x}\ell} - A_{(xy)\ell,\ell}\}&(a + b\ell)\\ &= [x\cdot y - 1\cdot xy,a] + \{b(xy - 1\cdot xy) - (yx - xy\cdot 1)b\}\ell\\ &= \{[x,y]b\}\ell\\ &= D_{B,[x,y]}(a + b\ell)\end{aligned}$$

so $D_{B,u}$ is strictly inner (even an associator derivation) with indicator $\frac{1}{\mu}\{[y\ell,\bar{x}\ell] - [(xy)\ell,\ell]\} = (xy - \overline{yx}) - (xy - \overline{xy}) = 0$. ■

We will see in Chapter VII that a Cayley algebra may be written as $\mathbb{C}(B,\mu)$ for lots of different quaternion subalgebras B, so there are lots of quaternion dilations and translations. Indeed, we will see any automorphism is a product of quaternion dilations and any derivation a sum of quaternion translations, so all automorphisms and derivations are inner.

For a split Cayley algebra we can explicitly show all derivations (into any bimodule) are inner. We turn now to a detailed investigation of derivations in the split case.

Exercises IV.3

3.1 Show $L_{(xy)\ell,\ell}\, L_{x^{-1}\ell,\, y^{-1}\ell}$ is an automorphism of $\mathbb{C}(B,\mu)$ by computing its indicator $z = \{R^{-1}_{(xy)\ell\cdot\ell}\, R_{(xy)\ell}\, R_{\ell}\}\, \{R^{-1}_{(x^{-1}\ell)(\bar{y}^{-1}\ell)}\, R_{x^{-1}\ell}\, R_{\bar{y}^{-1}\ell}\}\,(1)$.

3.2 Show $L_{1 + (x+\bar{y})\ell}\, T_{B,\, 1 + \mu xy} = L_{1 + x\ell}\, L_{1 + \bar{y}\ell}$ by computing directly the actions on $a + b\ell$ (when $yx = 0$).

3.3 Use a dual-numbers argument to derive the Quaternion Translation Thoerem 3.8 from the Quaternion Translation Theorem 3.1. Can you show the Trace Zero Quaternion Lemma 3.9 from the Norm One Quaternion Lemma 3.3?

3.4 Compute the action of $L_{\ell,x^2}\, L_{\ell,1}\, L_{x\ell,\ell}$ on $\mathbb{C}(B,\mu)$ ($x \in B$ invertible) and deduce it is an automorphism.

§4. Derivations of split Cayley algebras

In this section we display in concrete form the structure of the derivation algebra Der($\mathbb{C}$) of a split Cayley algebra over an arbitrary Φ. This turns out to be a split Lie algebra of type G_2. The fact that Cayley algebras give rise to exceptional Lie algebras is responsible for much of the attention given alternative algebras.

Out first job will be to prove all derivations are inner,

$$\mathrm{Der}(\mathbb{C}) = \mathrm{Inder}(\mathbb{C}) \qquad \mathrm{Outder}(\mathbb{C}) = 0.$$

Over a field Φ this could be proved by proving that Der($\mathbb{C}$) is simple and Inder($\mathbb{C}$) is a nonzero ideal therefore, Inder($\mathbb{C}$) must be all of Der($\mathbb{C}$) (see 0.00).

There is a more constructive approach, which works for arbitrary Φ and for arbitrary derivations into a bimodule.

4.1 (Split Inner Derivation Theorem) All derivations D of a split Cayley algebra $\mathbb{C}$ into a unital bimodule M are strict inner derivations with indicator zero,

$$(4.2) \qquad D = D_{e_1, m_2} + \sum_{i=1}^{3} A_{e_{12}^{(i)},\, m_{21}^{(i)}}$$

$$(m_2 = D(e_2),\ m_{21}^{(i)} = e_2\, D(e_{21}^{(i)})\, e_1, \Sigma\, e_{12}^{(i)} m_{21}^{(i)}$$

$$= \Sigma\, m_{21}^{(i)} e_{12}^{(i)} = 0).$$

If $1/3 \in \Phi$ all derivations are strictly standard,

$$(4.3) \qquad D = D_{e_1, m_2} - 1/3 \sum_{i=1}^{3} D_{e_{12}^{(i)},\, m_{21}^{(i)}}.$$

Proof. The alternate expression (4.3) follows from the basic formula (4.2) by (2.20).

So we turn to (4.2). Our first step is to subtract off an inner derivation from D so that the resulting derivation $\tilde{D}$ kills e_1 and e_2. This can be done quite generally: it's not true that a derivation kills idempotents, the way it kills units, but it can be persuaded to after a little straightening-out.

4.4 (Idempotent-Killing Lemma) If $D: A \to M$ is a derivation of an alternative algebra A into a bimodule M, where $e_1,\ldots,e_n$ are pairwise orthogonal idempotents in A, then

$$\tilde{D} = D + \Sigma_{i\leq j} D_{e_i,D(e_j)}$$

is a derivation $\tilde{D}$: $A \to M$ which kills all e_i, $\tilde{D}(e_i) = 0$, and consequently maps Peirce spaces $\tilde{D}(A_{ij}) \subset M_{ij}$.

Proof. To see $\tilde{D}$ kills e_k, simply compute $\tilde{D}(e_k)$ $= D(e_k) + \Sigma_{i\leq j} D_{e_i,D(e_j)} e_k = D(e_k) + \Sigma_{i\leq j} [[e_i,D(e_j)],e_k]$ $+ 3\Sigma_{i\leq j} [e_i,D(e_j),e_k]$. Of the three parts to the expression for $\tilde{D}(e_k)$, the first is $D(e_k)$, the last vanishes since $[e_i,M,e_j] = 0$, and the middle is $- D(e_k)$ since $\Sigma_{i\leq j}\{ e_iD(e_j) - D(e_j)e_i\} e_k - e_k\{e_iD(e_j) - D(e_j)e_i\}$ $= \Sigma_{i\leq j} \{e_iD(e_j)e_k + e_kD(e_j)e_i\} - \Sigma_{i\leq j} \{D(e_j)e_ie_k + e_ke_iD(e_j)\}$ (by the above associativity) $= \Sigma_{i\leq j} \{D(e_ie_j) - D(e_i)e_j\}e_k$ $+ e_k\{D(e_je_i) - e_jD(e_i)\} - \Sigma_{i\leq j} \{D(e_j)e_ie_k + e_ke_iD(e_j)\}$ (as $D(xy) = D(x)y + xD(y)$) $= \Sigma_i\{D(e_i)e_k + e_kD(e_i)\}$ $- \Sigma_{i\leq j=k} \{D(e_i)e_k + e_kD(e_i)\} - \Sigma_{k=i\leq j} \{D(e_j)e_k + e_kD(e_j)\}$ (by orthogonality of the e's) $= -\{D(e_k)e_k + e_kD(e_k)\} = - D(e_k^2) = - D(e_k)$. Thus the three expressions for $\tilde{D}(e_k)$ add up to zero.

Once $\tilde{D}$ kills all e_k we have $\tilde{D}(A_{ij}) = \tilde{D}(e_i A e_j)$ $= e_i \tilde{D}(A) e_j \subset e_i M e_j = M_{ij}$. ■

Applying this to our situation we see $\tilde{D} = D + D_{e_1, m_1} = D - D_{e_1, m_2}$ kills e_1 (hence also $e_2 = 1 - e_1$) for $m_2 = D(e_2) = D(1 - e_1) = -D(e_1) = -m_1$.

The formula (4.2) thus reduces to $\tilde{D} = \Sigma A_{e_{12}^{(i)}, m_{21}^{(i)}}$. Let us note at this point that

$$m_{21}^{(i)} = \tilde{D}(e_{21}^{(i)}).$$

Since $m_{21}^{(i)} = e_2 D(e_{21}^{(i)}) e_1$ and $\tilde{D}(e_{21}^{(i)}) = \tilde{D}(e_2 e_{21}^{(i)} e_1) = e_2 \tilde{D}(e_{21}^{(i)}) e_1$ if $\tilde{D}$ kills e_1 and e_2, this means $D(e_{21}^{(i)})$ and $\tilde{D}(e_{21}^{(i)})$ have the same component in M_{21}, or that their difference $D_{e_1, m_2}(e_{21}^{(i)})$ has zero component in M_{21}. But $D_{e_1, m_2}(e_{21}^{(i)}) = [[e_1, m_2], e_{21}^{(i)}]$ $- 3[e_1, m_2, e_{21}^{(i)}]$ where (since $m_2 = D(e_2) = e_2 \circ D(e_2) = e_2 \circ m_2$ implies $m_2 = m_{12} + m_{21}$ for $m_{ij} \in M_{ij}$) neither $[[e_1, m_2], e_{21}^{(i)}]$ $= [m_{12} - m_{21}, e_{21}^{(i)}] \in M_{12}A_{21} + M_{21}A_{21} + A_{21}M_{21} + A_{21}M_{12} \subset M_{11} + M_{12} + M_{22}$ nor $[m_2, e_1, e_{21}^{(i)}] = (m_2 e_1) e_{21}^{(i)} = m_{21} e_{21}^{(i)} \in M_{21}A_{21} \subset M_{12}$ has any component in M_{21}.

Replacing D by $\tilde{D}$, we must show that if D kills e_1 and e_2 it has the form

(4.2') $\quad D = \Sigma A_{e_{12}^{(i)}, m_{21}^{(i)}} \quad (m_{21}^{(i)} = D(e_{21}^{(i)}),\ \Sigma\, e_{12}^{(i)} m_{21}^{(i)}$
$\qquad\qquad = \Sigma\, m_{21}^{(i)} e_{12}^{(i)} = 0).$

We begin by showing orthogonality of e's and m's:

$$\begin{aligned} 0 = D(e_1) &= D(e_{12}^{(1)} (e_{12}^{(2)} e_{12}^{(3)})) \\ &= D(e_{12}^{(1)})\{e_{12}^{(2)} e_{12}^{(3)}\} + e_{12}^{(1)}\{D(e_{12}^{(2)}) e_{12}^{(3)}\} \\ &\quad + e_{12}^{(1)}\{e_{12}^{(2)}\ D(e_{12}^{(3)})\} = \Sigma_{i=1}^{3}\ D(e_{12}^{(i)})\{e_{12}^{(i+1)} e_{12}^{(i+2)}\} \end{aligned}$$

(by Permuting Formula V.3.9)

$$= -\Sigma e_{12}^{(i)} D(e_{12}^{(i+1)} e_{12}^{(i+2)}) \quad (\text{as } D(e_{12}^{(i)} \cdot e_{12}^{(i+1)} e_{12}^{(i+2)}) = D(e_1) = 0)$$

$$= -\Sigma e_{12}^{(i)} D(e_{21}^{(i)})$$

$$= -\Sigma e_{12}^{(i)} m_{21}^{(i)}.$$

Similarly $D((e_{12}^{(1)} e_{12}^{(2)}) e_{12}^{(3)}) = 0$ leads to $\Sigma m_{21}^{(i)} e_{12}^{(i)} = 0$.

This orthogonality implies in particular $\Sigma [e_{12}^{(i)}, m_{21}^{(i)}] = 0$ so $D' = \Sigma A_{e_{12}^{(i)}, m_{21}^{(i)}}$ is a derivation by the Associator Derivation Criterion 2.14. To finish (4.2') we need only show the derivations D and D' agree on the generators $e_{21}^{(j)}$ $(j = 1,2,3)$ since then they agree everywhere. But

$$D'(e_{21}^{(j)}) = \Sigma_i [e_{12}^{(i)}, m_{21}^{(i)}, e_{21}^{(j)}] = \Sigma_i [e_{21}^{(j)}, e_{12}^{(i)}, m_{21}^{(i)}]$$

$$= \Sigma_i (e_{21}^{(j)} e_{12}^{(i)}) m_{21}^{(i)} - e_{21}^{(j)} \Sigma_i e_{12}^{(i)} m_{21}^{(i)}$$

$$= \Sigma_i \delta_{ji} e_2 m_{21}^{(i)} = m_{21}^{(j)} \quad (\text{since } m_{21}^{(i)} \in M_{21})$$

$$= D(e_{21}^{(j)}).$$

This completes (4.2') and the theorem. ∎

Once we have this for the split case, we can use a standard field extension argument to get it for the general case.

4.5 (Inner Derivation Theorem) Any derivation of a Cayley algebra A over a field Φ into a unital bimodule M is strictly inner,

(4.6) $$D = \Sigma D_{x_i, m_i} + \Sigma A_{y_i, n_i} \quad (\Sigma y_i n_i = \Sigma n_i y_i = 0).$$

(4.7) $$D = \Sigma D_{z_i, w_i} \quad (\text{if char.}\Phi \neq 3).$$

Proof. As we noted in I.24, we can choose an extension field $\Omega \supset \Phi$ so that $A_\Omega = \mathbb{C}(\Omega)$ is split Cayley over Ω. $D_\Omega = D \otimes 1$ remains a derivation $A_\Omega = A \otimes \Omega \to M \otimes \Omega = M_\Omega$, so by the split case D_Ω is strictly inner: $D_\Omega = \Sigma\, D_{x_i,m_i} + \Sigma\, A_{y_i,n_i}$ where $\Sigma\, y_i n_i = \Sigma\, n_i y_i = 0$. From the bilinearity of $D_{x,m}$ and $A_{y,n}$ and yn and ny we can (by expressing $x,y \in A$ as Ω-linear combinations of $a,b \in A$ and moving the scalars to the m's and n's, eg. $D_{\omega a,m} = D_{a,\omega m}$) assume all x_i, y_i belong to A and all m_i, n_i to M_Ω. Choosing a basis $\{\omega_\alpha\}$ for Ω/Φ with $\omega_o = 1$ and writing $m_i = \Sigma m_{i\alpha} \otimes \omega_\alpha$, $n_i = \Sigma\, n_{i\alpha} \otimes \omega_\alpha$ for $m_{i\alpha}, n_{i\alpha} \in M$, we can identify coefficients of ω_o in $D \otimes \omega_0 = D_\Omega$ $= \Sigma_\alpha\{\Sigma_i\, D_{x_i,m_{i\alpha}} + \Sigma_i\, A_{y_i,n_{i\alpha}}\} \otimes \omega_\alpha$ to get $D = \Sigma_i\, D_{x_i,m_{io}} + \Sigma_i\, A_{y_i,n_{io}}$. Furthermore, identifying coefficients of ω_o in $0 = \Sigma\, y_i n_i$ $= \Sigma_\alpha\{\Sigma_i\, y_i n_{i\alpha}\} \otimes \omega_\alpha$ shows $\Sigma\, y_i n_{io} = 0$, and similarly $\Sigma\, n_{io} y_i = 0$. This establishes (4.6).

Once we have (4.6) we derive (4.7) as before. ∎

We have similar results for quaternion algebras

4.8 (Split Inner Derivation Theorem) All derivations of a split quaternion algebra Q into a unital bimodule are inner derivations,

(4.9) $$D = D_{e_{11},m} - D_{m_{11}} + A_{e_{12},m_{21}} + A_{e_{21},m_{12}}$$

$$(m = D(e_{22}),\ m_{ij} = e_{ii}\, D(e_{ij})e_{jj}, m_{ii} = e_{ij} m_{ji} = -m_{ij} e_{ji}).$$

If M is a regular bimodule then

(4.9a) $$D = D_{e_{11},m} - D_{m_{11}}$$

while if M is a Cayley bimodule then

(4.9b) $\quad D = D_{e_{11},m} + A_{e_{12},m_{21}} + A_{e_{21},m_{12}} \quad (e_{ij}m_{ji} = m_{ij}e_{ji} = 0)$

is strictly inner with indicator zero.

Proof. Just as in the Cayley case, $\tilde{D} = D + D_{e_{11},D(e_{11})}$ $= D - D_{e_{11},m_2}$ $(m_2 = -D(e_{11}) = D(e_{22}))$ kills e_{11} and hence also $e_{22} = 1 - e_{11}$, and $\tilde{D}(e_{ij}) = e_{ii}D(e_{ij})e_{jj} = m_{ij}$. Replacing D by $\tilde{D}$, it suffices to assume from the start $D(e_{ii}) = 0$.

In this case we claim D and $D' = -D_{m_{11}} + A_{e_{12},m_{21}} + A_{e_{21},m_{12}}$ agree on the basis $\{e_{11},e_{12},e_{21},e_{22}\}$ for Q and hence $D = D'$. Note $D'(e_{ij}) = -[m_{11},e_{ij}] + [e_{12},m_{21},e_{ij}] + [e_{21},m_{12},e_{ij}]$. If $i=j$ the associators vanish by Peirce Associativity V.3.6 (distinct Peirce spaces), and trivially the commutator $[m_{11},e_{11}]$ does too, so $D'(e_{ii}) = 0 = D(e_{ii})$. If $i \neq j$ the associators add up to $[e_{ij},m_{ji},e_{ij}] + [e_{ji},m_{ij},e_{ij}] = [e_{ji},m_{ij},e_{ij}]$ $= [e_{ij},e_{ji},m_{ij}] = (e_{ij}e_{ji})m_{ij} - e_{ij}(e_{ji}m_{ij}) = e_{ii}m_{ij} + e_{ij}(m_{ji}e_{ij})$ (observe $0 = D(e_{ji}e_{ij}) = D(e_{ji})e_{ij} + e_{ji}D(e_{ij}) = m_{ji}e_{ij} + e_{ji}m_{ij}$) $= m_{ij} + e_{ij}m_{ji}e_{ij}$, while for $(i,j) = (1,2)$ the commutator gives $-[m_{11},e_{12}] = -m_{11}e_{12} = -e_{12}m_{21}e_{12}$ and for $(i,j) = (2,1)$ gives $-[m_{11},e_{21}] = + e_{21}m_{11} = -e_{21}m_{12}e_{21}$; thus $D'(e_{ij})$ $= -e_{ij}m_{ji}e_{ij} + m_{ij} + e_{ij}m_{ji}e_{ij} = m_{ij} = D(e_{ij})$. So $D = D'$.

In a regular bimodule all associators vanish, and (4.9) reduces to (4.9a). In a Cayley bimodule (see I.7.4) we have $m_{ii} = e_{ij}m_{ji} = -m_{ij}e_{ji} = (e_{ii}e_{ij})(m_{ji}e_{ii}) \in e_{ii}Me_{ii}$ (Middle Moufang) $= e_{ii}(Q\ell)e_{ii} = (Qe_{ii}\overline{e_{ii}})\ell = 0$ so (4.9) reduces to (4.9b). ∎

WARNING: In contrast to the Cayley case, not all derivations of a quaternion algebra are strictly inner. The trouble comes with the regular bimodule; in (4.9a) we can't get rid of the $D_{m_{11}}$. In this case $e_{12}\circ m_{21} = m_{11} - m_{22}$ is in the center (see Ex. 5.2), so $D_{m_{11}} = D_{m_{11}-m_{22}} + D_{m_{11}+m_{22}} = D_{m_{11}+m_{22}} = D_{[e_{12},m_{21}]}$ $= D_{e_{12},m_{21}}$ by associativity, so in characteristic $\neq 2$ we can replace (4.9a) by

(4.9a') $$D = D_{e_{11},m} - \frac{1}{2} D_{e_{12},m_{21}}$$

in the split case, and

(4.9a'') $$D = \Sigma\, D_{x_i,m_i}$$

in general. <u>Thus D is strictly inner except in characteristic 2.</u> $\blacksquare$

The same field-extension argument as before shows

4.10 (Inner Derivation Theorem) Any derivation of a quaternion algebra over a field Φ into a unital bimodule is an inner derivation,

(4.11) $$D = D_m + \Sigma\, A_{x_i,m_i} + \Sigma\, D_{y_i,n_i}$$

where $m = \Sigma\, x_i m_i = \Sigma\, m_i x_i = 0$ (and D is strictly inner) for a Cayley bimodule and $m_i = 0$ for a regular bimodule. $\blacksquare$

Multiplication Tables

It will be useful to have the following formulas for standard derivations:

(4.12) $$D_{1,x} = 0$$

(4.13) $$D_{x,x} = 0, \qquad D_{x,y} + D_{y,x} = 0$$

(4.14) $$D_{xy,z} + D_{yz,x} + D_{zx,y} = 0.$$

Here (4.12) is trivial since $[1,A] = [1,A,A] = 0$, and (4.13) follows from alternativity $[x,x] = [x,x,A] = 0$, and from linearization. The only hard part is (4.14). On the one hand, since D_z is linear in the variable z and since by II.2.11 $[xy,z] + [yz,x] + [zx,y] = 3[x,y,z]$, we have

$$D_{[xy,z]} + D_{[yz,x]} + D_{[zx,y]} = 3D_{[x,y,z]}.$$

On the other hand $[xy,z,a] + [yz,x,a] + [zx,y,a] = \{[xy,z,a] - [x,yz,a] + [x,y,za]\} - \{[x,y,za] + [a,y,zx]\}$ (by alternativity) $= \{x[y,z,a] + [x,y,z]a\} - \{a[x,y,z] + x[a,y,z]\}$ (associator identity III.2.4 and left bumping) $= [x,y,z]a - a[x,y,z]$ or

$$A_{xy,z} + A_{yz,x} + A_{zx,y} = D_{[x,y,z]}.$$

If we multiply this by 3 and subtract from the previous equation we get (4.14), recalling the definition $D_{x,y} = D_{[x,y]} - 3A_{x,y}$.

4.15 Remark. The importance of these is in cutting down the number of standard derivations one has to consider. If A has dimension n with basis $x_1,\ldots,x_n$ there are n^2 standard D_{x_i,y_i}; (4.12) reduces this to $(n-1)^2$, (4.13) to $\frac{(n-1)(n-2)}{2}$, and (4.14) to $\frac{2}{3}\frac{(n-1)(n-2)}{2} = \frac{(n-1)(n-2)}{3}$. For example, in the Cayley algebra $A = \mathbb{C}$ of dimension 8, instead of $8^2 = 64$ derivations there are really only $\frac{7\cdot 6}{3} = 14$. ∎

In order to graphically illustrate the action of the derivation algebra $Der(\mathfrak{C})$ of a split Cayley algebra $\mathfrak{C}$ over an arbitrary Φ we form the table below. Note

$$L_{e_2} = I - L_{e_1}, R_{e_2} = I - R_{e_1}, D_{e_2} = -D_{e_1}.$$

For typographical convenience we abbreviate $e_{12}^{(i)}$ by f_i and $e_{21}^{(i)}$ by g_i so $e_1, e_2, f_1, f_2, f_3, g_1, g_2, g_3$ forms a basis for $\mathfrak{C}$ (see V.6 for the resulting multiplication table for $\mathfrak{C}$).

4.16 Action Table for L_x, R_x, D_x

	$L(e_1)$	$R(e_1)$	$D(e_1)$	$L(f_i)$	$R(f_i)$	$D(f_i)$	$L(g_i)$	$R(g_i)$	$D(g_i)$
e_1	e_1	e_1	0	0	f_i	$-f_i$	g_i	0	g_i
e_2	0	0	0	f_i	0	f_i	0	g_i	$-g_i$
f_i	f_i	0	f_i	0	0	0	e_2	e_1	e_2-e_1
f_{i+1}	f_{i+1}	0	f_{i+1}	g_{i+2}	$-g_{i+2}$	$2g_{i+2}$	0	0	0
f_{i+2}	f_{i+2}	0	f_{i+2}	$-g_{i+1}$	g_{i+1}	$-2g_{i+1}$	0	0	0
g_i	0	g_i	$-g_i$	e_1	e_2	e_1-e_2	0	0	0
g_{i+1}	0	g_{i+1}	$-g_{i+1}$	0	0	0	$-f_{i+2}$	f_{i+2}	$-2f_{i+2}$
g_{i+2}	0	g_{i+2}	$-g_{i+2}$	0	0	0	f_{i+1}	$-f_{i+1}$	$2f_{i+1}$

From this we compute the $A_{x,y} = L_{xy} - L_xL_y = [L_x, R_y]$. Note

$$A_{e_2,x} = -A_{e_1,x}; A_{x,x} = 0; A_{x,y} = -A_{y,x}.$$

4.17 Action Table for $A_{x,y}$

	$A(e_1,f_i)$	$A(e_1,g_i)$	$A(f_i,f_{i+1})$	$A(g_i,g_{i+1})$	$A(f_i,g_i)$	$A(f_i,g_{i+1})$	$A(f_i,g_{i+2})$
e_1	0	0	g_{i+2}	f_{i+2}	0	0	0
e_2	0	0	$-g_{i+2}$	$-f_{i+2}$	0	0	0
f_i	0	0	0	$-g_{i+1}$	0	0	0
f_{i+1}	g_{i+2}	0	0	g_i	f_{i+1}	$-f_i$	0
f_{i+2}	$-g_{i+1}$	0	e_2-e_1	0	f_{i+2}	0	$-f_i$
g_i	0	0	$-f_{i+1}$	0	0	g_{i+1}	g_{i+1}
g_{i+1}	0	f_{i+2}	f_i	0	$-g_{i+1}$	0	0
g_{1+2}	0	$-f_{i+1}$	0	e_2-e_1	$-g_{i+2}$	0	0

Since $D_{x,y} = Ad_{[x,y]} - 3A_{x,y}$ we can put the two tables together to get one for standard inner derivations. Note

$$[e_1,f_i] = f_i, \ [e_1,g_i] = -g_i, \ [f_i,g_i] = \delta_{ij}(e_1-e_2),$$

$$D_{e_2,x} = -D_{e_1,x}, D_{x_{12},y_{12}} = D_{e_1,x_{12}y_{12}}, D_{x_{12},y_{21}} = D_{e_2,x_{21}y_{21}}$$

since $D_{1,x} = 0$ and $0 = D_{e_1x_{12},y_{12}} + D_{x_{12}y_{12},e_1} + D_{y_{12}e_1,x_{12}}$ (by (4.14)) $= D_{x_{12},y_{12}} - D_{e_1,x_{12}y_{12}}$ and similarly $D_{x_{21},y_{21}}$ $= D_{e_{21}x_{21}y_{21}}$.

4.18 Action Table for $D_{x,y}$

	$D(e_1,f_i)$	$D(e_2,g_i)$	$D(f_i,g_i)$	$D(f_i,g_{i+1})$	$D(f_i,g_{i+2})$
e_1	$-f_i$	g_i	0	0	0
e_2	f_i	$-g_i$	0	0	0
f_i	0	e_2-e_1	$2f_i$	0	0
f_{i+1}	$-f_{i+2}$	0	$-f_{i+1}$	$3f_i$	0
f_{i+2}	g_{i+1}	0	$-f_{i+2}$	0	$3f_i$
g_i	e_1-e_2	0	$-2g_i$	$-3g_{i+1}$	$-3g_{i+2}$
g_{i+1}	0	f_{i+2}	g_{i+1}	0	0
g_{i+2}	0	$-f_{i+1}$	g_{i+2}	0	0