§3 From projective to affine and back again

We will show how to construct affine planes from projective planes and vice versa, and that (in a suitable sense) the concepts are equivalent: if you've got one you've got the other (but not quite).

Given a pair (Π, L_∞) consisting of a projective plane and a distinguished <u>line at infinity</u> L_∞ , we construct an affine plane

$$\mathrm{Aff}(\Pi, L_\infty) = (P_a, L_a, I_a)$$

(3.1)

$$L_a = L \setminus L_\infty \qquad \text{(remove one line)}$$

$$P_a = P \setminus P(L_\infty) \qquad \text{(remove all points on that line)}$$

$$I_a = I \cap (P_a \times L_a) \qquad \text{(take induced incidence).}$$

We call this the <u>affine restriction</u> of Π relative to L_∞ . Notice that two lines are parallel in $\mathrm{Aff}(\Pi, L_\infty)$ iff their intersection (which always exists in Π) does not belong to the affine part of Π, ie iff the lines intersect on the line at infinity L_∞ :

$$L \parallel M \ \text{ iff } \ L \wedge M \in P(L_\infty) \ .$$

Note also that $P_a(L) = P(L) \setminus L \wedge L_\infty$ for $L \in L_a$, so the affine line $L \in L_a$ is not quite the same as the projective line $L \in L$ as far as its points go.

3.2 (Affine Restriction Theorem) If Π is a projective plane then for any line L_∞ the affine restriction $\Pi_a = \mathrm{Aff}(\Pi, L_\infty)$ is an affine plane.

Proof. Aff I: 2 points $P, Q \in P_a$ lie on a unique line $P \vee Q$ since $P \vee Q$ (which exists by Proj. I) is not L_∞ and hence belongs to L_a .

Aff II: two lines L, L' in L_a either intersect in P_a (if $L \wedge L' \notin P(L_\infty)$) or are disjoint (if $L \wedge L' \in P(L_\infty)$) by Proj. II.

Aff III: given $L \in L_a$ (so $L \neq L_\infty$) and $P \in P_a$ (so $P \notin P(L_\infty)$) there is a unique point of intersection $L \wedge L_\infty = Q$ in P (Proj. II) and a unique line $L' = P \vee Q$ in L(Proj. I). Since $P \notin P(L_\infty)$ we have $L' \neq L_\infty$, so $L' \in L_a$ and $P \in P(L')$ for $L' \| L$. Such an L' is unique, since any other L'' on P parallel to L must intersect L on L_∞ , hence at Q, and so $L'' = P \vee Q = L'$.

Aff IV: Pick a line $L \neq L_\infty$ (we know there are at least 3 lines by 1.10*), and points P_1, P_2 on L different from $L \wedge L_\infty$ (L has at least 3 points by 1.10). By 1.6 there is a point P_3 off $L = P_1 \vee P_2$ and off L_∞ , so P_1, P_2, P_3 are non-collinear affine points. ■

This construction is <u>functorial</u>, from the category of projective planes with line at infinity (a morphism $(\Pi, L_\infty) \to (\tilde{\Pi}, \tilde{L}_\infty)$ in this category being an isomorphism $\Pi \to \tilde{\Pi}$ of planes which sends L_∞ into $\tilde{L}_\infty$) to the category of affine planes,

$$\text{Projective planes with line at infinity} \xrightarrow{\text{Affine restriction}} \text{Affine planes.}$$

Indeed, if $\Pi \xrightarrow{\sigma} \tilde{\Pi}$ is an isomorphism sending $L_\infty \to \tilde{L}_\infty$ then we obtain an isomorphism $\mathrm{Aff}(\sigma)$ of affine planes by restriction to $\mathrm{Aff}(\Pi)$: $\{\mathrm{Aff}(\sigma)\}(L) = \sigma(L), \{\mathrm{Aff}(\sigma)\}(P) = \sigma(P)$ for $L \in L_a$, $P \in P_a$. This restriction does indeed map onto $\mathrm{Aff}(\tilde{\Pi})$: $\sigma(L) \in \tilde{L}_a$, ie $\sigma(L) \neq \tilde{L}_\infty$, and $\sigma(P) \in \tilde{P}_a$, ie $\sigma(P) \notin P(\tilde{L}_\infty)$ by bijectivity of σ. It certainly preserves incidence in $\mathrm{Aff}(\Pi)$ since it did in Π. Clearly $\mathrm{Aff}(1) = 1$ and $\mathrm{Aff}(\sigma \circ \tau) = \mathrm{Aff}(\sigma) \circ \mathrm{Aff}(\tau)$ by restriction, using $\mathrm{Aff}(\sigma)(P_a) = \tilde{P}_a$ and $\mathrm{Aff}(\sigma)(L_a) = \tilde{L}_a$.

In short, we have a natural way of constructing affine planes from projective planes by means of affine restriction.

Now start with any old affine plane $\Pi_a = (P_a, L_a, I_a)$. We construct a projective completion

$$\mathrm{Proj}(\Pi_a) = (P,L,I)$$

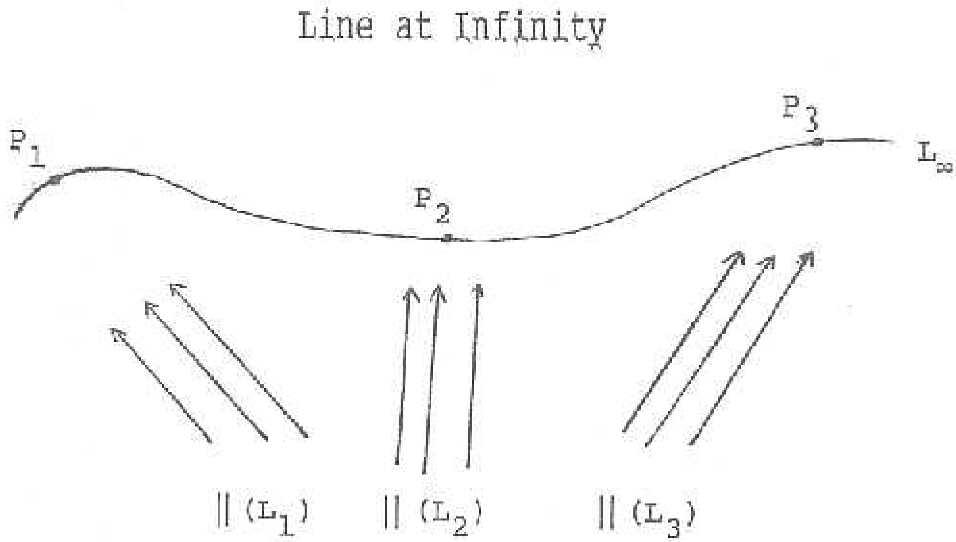
(3.4)

$$L = L_a \cup L_\infty \qquad \text{(adjoin an ideal line)}$$

$$P = P_a \cup \|(\Pi_a) \qquad \text{(adjoin one ideal point for each parallel class)}$$

$$\begin{aligned} I &= I_a \cup I_\infty \cup I_{\|} \\ &= I_a \cup \{\mathrm{all}(\|(L), L_\infty)\} \cup \{\mathrm{all}(\|(L), L)\}. \end{aligned}$$

Thus the ordinary points $P \in L_a$ lie on the same lines as before, while the ideal points $\|(L)$ lie on L_∞ and all lines L' parallel to L. Observe that two ordinary lines intersect on L_∞ (are incident to some $P = \|(L)$ on L_∞) iff they are parallel in Π_a (belong to the same parallel class $\|(L)$).



Projective Completion

3.5 (Completion Theorem) If Π_a is an affine plane then its projective completion $\Pi = \mathrm{Proj}(\Pi_a)$ is a projective plane.

Proof. Proj. I: if $P, P' \in P_a$ are ordinary points they do not lie on L_∞, and in view of I_a they lie on a unique affine line $P \vee P'$ (Aff I). If $P = ||(L)$, $P' = ||(L')$ are ideal points they lie on L_∞ but in view of $I_{||}$ on no ordinary line (if they lie on M then $L || M || L'$, implying $P = ||(L) = ||(L') = P'$). If P is ordinary but $P' = ||(L)$ is ideal then in view of $I_{||}$ the only line they lie on is $P \vee L$ (Aff III) ($P \notin P(L_\infty)$ and if P, P' are on an ordinary M then $M || L$).

Proj. II: If L, L' are ordinary lines which are not parallel then by Aff III their intersection is the unique ordinary point $L \wedge L'$ in view of I_a (by $I_{||}$ they would only have an ideal point $||(M)$ in common if $L||L'||M$). If L, L' are ordinary but parallel they have no ordinary intersection, and $L \wedge L' = ||(L) = ||(L')$ as their unique ideal intersection by $I_{||}$. If L is ordinary but $L' = L_\infty$ is ideal their unique intersection is $L \wedge L_\infty = ||(L)$ (they clearly have no ordinary intersection, and the only ideal point on L is $||(L)$ by $I_{||}$).

Proj. III: 4-points exist in Π because they already exist in Π_a by the Parallelogram Lemma 2.2. ■

This construction is functorial: any isomomorphism $\Pi_a \xrightarrow{\sigma} \tilde{\Pi}_a$ induces an isomomorphism $\Pi \xrightarrow{Proj(\sigma)} \tilde{\Pi}$ by

$$\begin{aligned} Proj(\sigma)(P) &= \sigma(P) && P \in P_a \\ Proj(\sigma)(||(L)) &= \tilde{||}\sigma(L) && L \in L_a \\ Proj(\sigma)(L) &= \sigma(L) && L \in L_a \\ Proj(\sigma)(L_\infty) &= \tilde{L}_\infty \end{aligned}$$

(note $L||L' \Rightarrow \sigma(L)\tilde{||}\sigma(L')$, so $\tilde{||}\,\sigma(L)$ depends only on $||(L)$). Certainly $Proj(\sigma)$ is bijective on points and lines. It preserves incidence I: if P, L are ordinary then $P\ I\ L \Leftrightarrow P\ I_a\ L \Leftrightarrow \sigma(P)\ \tilde{I}_a\ \sigma(L) \Leftrightarrow \sigma(P)\ \tilde{I}\ \sigma(L)$, (so $Proj(\sigma)$ preserves I_a), if $P = ||(L)$ is ideal then $||(L)\ I\ L_\infty$ and $\tilde{||}\,\sigma(L)\ \tilde{I}\ \tilde{L}_\infty$ (so $Proj(\sigma)$ preserves I_∞), and if $P = ||(M)$ is ideal, L ordinary then $P\ I\ L \Leftrightarrow M||L \Leftrightarrow \sigma(M)\tilde{||}\sigma(L) \Leftrightarrow \tilde{||}\,\sigma(M)\ \tilde{I}\ \sigma(L)$ (so $Proj(\sigma)$ preserves $I_{||}$). Clearly $Proj(1) = 1$ and $Proj(\sigma \circ \tau) = Proj(\sigma) \circ Proj(\tau)$ by the form of $Proj(\sigma)$.

Notice that $\mathrm{Proj}(\sigma)$ preserves the line at infinity, so we can view Proj as a functor

$$\text{Affine planes} \xrightarrow{\mathrm{Proj}} \text{Projective planes with line at infinity.}$$

In short, we have a natural way of constructing projective planes (with distinguished lines at infinity) out of affine planes by means of projective completion.

3.6 (Equivalence Theorem) Affine restriction and projective completion are reciprocal functors between the categories of affine planes and projective planes with lines at infinity.

More precisely, if Π_a is an affine plane then the affine restriction of its projective completion Π relative to L_∞ is just Π_a

$$\mathrm{Aff}(\mathrm{Proj}(\Pi_a)) = \Pi_a \ ,$$

while if Π is a projective plane and L_∞ a line then the projective completion of its affine restriction Π_a relative to L_∞ is canonically isomorphic to Π ,

$$(\Pi,\ L_\infty) \cong \mathrm{Proj}(\mathrm{Aff}(\Pi,\ L_\infty))$$

under the isomorphism

$$\begin{aligned}
\sigma(P) &= P & &(P \in \mathcal{P} \setminus \mathcal{P}(L_\infty)) \\
\sigma(P_\infty) &= \|(0 \vee P_\infty) & &(P_\infty \in \mathcal{P}(L_\infty)) \\
\sigma(L) &= L & &(L \in \mathcal{L} \setminus L_\infty) \\
\sigma(L_\infty) &= \tilde{L}_\infty \ .
\end{aligned}$$

Proof. It's easy to show Aff $\circ$ Proj is the identity functor. Let Π_a be an affine plane, (Π, L_∞) its completion, and $\tilde{\Pi}_a = \mathrm{Aff}(\Pi, L_\infty)$. Then $\Pi_a = (P_a, L_a, I_a)$, $\Pi = (P,L,I)$ for $L = L_a \cup L_\infty$, $P = P_a \cup \{||(L)\}$, $I = I_a \cup I_\infty \cup I_{||}$, and $\tilde{\Pi}_a = (\tilde{P}_a, \tilde{L}_a, \tilde{I}_a)$ for $\tilde{L}_a = L \setminus L_\infty = L_a$, $\tilde{P}_a = P \setminus P(L_\infty) = P_a$ (since by I_∞ the points on L_∞ are precisely the $||(L)$, ie precisely the ideal points of Π), so $\tilde{I}_a = I \cap (\tilde{P}_a \times \tilde{L}_a)$ $= I \cap (P_a \times L_a) = I_a$ (since $I = I_a \cup I_\infty \cup I_{||}$).

Now assume we are given a projective plane Π with distinguished line L_∞. Its associated affine plane is Π_a $= \mathrm{Aff}(\Pi, L_\infty) = (P_a, L_a, I_a) = (P \setminus P(L_\infty), L \setminus L_\infty, I \cap P_a \times L_a)$, which has completion $\tilde{\Pi} = (\tilde{P}, \tilde{L}, \tilde{I})$ for

$$\tilde{P} = P_a \cup ||(L_a)$$

$$\tilde{L} = L_a \cup \tilde{L}_\infty$$

$$\tilde{I} = I_a \cup \tilde{I}_\infty \cup \tilde{I}_{||}.$$

Here $L = L_a \cup L_\infty \xrightarrow{\sigma} L_a \cup \tilde{L}_\infty = \tilde{L}$ is clearly a bijection on lines, and $P = P_a \cup P(L_\infty) \xrightarrow{\sigma} P_a \cup \{||(L)\} = \tilde{P}$ is a bijection on points since $P_\infty \longleftrightarrow O \vee P_\infty \longleftrightarrow ||(O \vee P_\infty)$ is a bijection $P(L_\infty) \longleftrightarrow ||(L_a)$ ($P_\infty \to O \vee P_\infty$, $L \to L \wedge L_\infty$ are inverse bijections $P(L_\infty) \longleftrightarrow L(O)$, and $L \longrightarrow ||(L)$, $||(L) \to L \vee O$ are inverse bijections $L(O) \longleftrightarrow ||(L_a)$). This preserves incidence since $I = I_a \cup I_\infty \cup I_{||}$ where under σ $I_a \to I_a$, $I_\infty = P(L_\infty) \times L_\infty \to ||(L_a) \times \tilde{L}_\infty = \tilde{I}_\infty$, and $I_{||} = I \cap \{P(L_\infty) \times L_a\} = \{(P_\infty, L) \mid P_\infty \in P(L_\infty), L \in L_a, P_\infty = L \wedge L_\infty\}$ $\longrightarrow \{(||(O \vee P_\infty), L) \mid P_\infty = L \wedge L_\infty\} = \{(||(L), L)\} = \tilde{I}_{||}$ (note

$L \wedge (0 \vee P_\infty) = P_\infty \Longrightarrow L \| (0 \vee P_\infty) \Longrightarrow \|(L) = \|(0 \vee P_\infty))$.
Thus σ is an isomorphism of planes (and takes the distinguished line L_∞ into the distinguished line $\tilde{L}_\infty$). ∎

This shows, for example, that every affine plane Π_a can be obtained from a projective plane by affine restriction, $\Pi_a = \mathrm{Aff}(\Pi, L_\infty)$ where $(\Pi, L_\infty) = \mathrm{Proj}(\Pi_a)$. This projective plane is unique up to isomorphism: if $\mathrm{Aff}(\Pi, L_\infty) \cong \mathrm{Aff}(\tilde{\Pi}, \tilde{L}_\infty)$ then $(\Pi, L_\infty) \cong \mathrm{Proj}(\mathrm{Aff}(\Pi, L_\infty)) \cong \mathrm{Proj}(\mathrm{Aff}(\tilde{\Pi}, \tilde{L}_\infty)) \cong (\tilde{\Pi}, \tilde{L}_\infty)$.

However, be careful to note that if we start with an affine plane Π_a , complete it to a projective plane Π by adding a line L_∞ , then delete from Π a <u>different</u> line $L \neq L_\infty$, the resulting affine plane $\Pi_a' = \Pi \setminus L$ need not look like the Π_a we began with. One must be careful to take away just what one added.

Similarly every projective plane Π can be obtained up to isomorphism from an affine plane by projective completion, $\Pi \cong \mathrm{Proj}(\Pi_a)$ where $\Pi_a = \mathrm{Aff}(\Pi, L_\infty)$ for any line L_∞ of Π. But this affine plane is <u>not</u> unique! We can get Π by throwing out <u>any</u> line L (getting an affine $\mathrm{Aff}(\Pi, L) = \Pi \setminus L$) and then putting it back in again, so Π is the completion of various affine $\Pi \setminus L$'s but there is no reason why these $\Pi \setminus L$'s should look alike. In general, a projective plane looks different when viewed from different lines L and L'; equivalently, there will generally not be an isomorphism of the whole plane sending L into L'. (Note

that if $\mathrm{Aff}(\Pi, L) \cong \mathrm{Aff}(\Pi, L')$ then
$(\Pi, L) \cong \mathrm{Proj}(\mathrm{Aff}(\Pi, L)) \cong \mathrm{Proj}(\mathrm{Aff}(\Pi, L') \cong (\Pi, L')$, ie there is an isomorphism $\Pi \xrightarrow{\sigma} \Pi$, $L \xrightarrow{\sigma} L'$).

Our natural correspondences were between affine planes and projective planes <u>with distinguished line</u>; a projective plane alone can give rise to very different choices of line at infinity.