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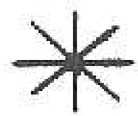
Appendix V

Alternative Algebras in Geometry

§1 Projective planes

A plane (or incidence plane) $\Pi = (P, L, I)$ consists of a set P of points, a set L of lines, and an incidence relation $I \subset P \times L$. When $P \in P$ and $L \in L$ are I-related, $(P,L) \in I$, we write PIL and say "P is incident to L", "P is on L", "P lies on L", "L lies on P", "L goes through P", "L is incident to P", etc. A collection of points are collinear if they lie on a common line, and a collection of lines are concurrent if they lie on a common point:

Collinear



Concurrent .

A homomorphism $\Pi \xrightarrow{\sigma} \tilde{\Pi}$ of planes consists of a mapping $P \xrightarrow{\sigma_p} \tilde{P}$ of points and a mapping $L \xrightarrow{\sigma_\ell} \tilde{L}$ of lines which preserve incidence,

$$P\ I\ L \Longrightarrow \sigma_p(P)\ \tilde{I}\ \sigma_\ell(L) .$$

We will consistently denote points by letters P,Q,R, etc. and lines by L,M,N, etc. Therefore it will cause no confusion if we use the same symbol σ for the map on points, the map on lines, and the homomorphism of planes (though a purist would object, for it could happen that $P \cap L \neq \phi$ and $\sigma_p \neq \sigma_\ell$ on $P \cap L$) . We can write the homomorphism condition as $(\sigma \times \sigma)\ I \subset \tilde{I}$.

As always, the composition of two homomorphisms

$\Pi \xrightarrow{\sigma} \tilde{\Pi} \xrightarrow{\tau} \tilde{\tilde{\Pi}}$ is again a homomorphism, and the identity map on points and lines is a homomorphism 1_Π of Π into itself.

A homomorphism $\Pi \xrightarrow{\sigma} \tilde{\Pi}$ is an <u>isomorphism</u> if it has an <u>inverse</u> homomorphism $\tilde{\Pi} \xrightarrow{\tau} \Pi$, $\tau \circ \sigma = 1_\Pi$ and $\sigma \circ \tau = 1_{\tilde{\Pi}}$. In this case τ coincides with the set-theoretic inverse of σ on points and lines, so σ is an isomorphism iff it is bijective on points and lines and

$$P \; I \; L \iff \sigma(P) \; \tilde{I} \; \sigma(L) \; .$$

The product of two isomorphisms is again an isomorphism. An isomorphism of a plane Π onto itself is called an <u>automorphism</u> (more commonly a <u>collineation</u>), and an isomorphism of a plane onto its dual Π^* is called an <u>antiautomorphism</u> (more commonly, <u>correlation</u>).

It must be stressed that points and lines are completely arbitrary sorts of objects - a line doesn't have to consist of points. We denote the points on a line L by

$$P(L) = \{P \in P \mid P \; I \; L\}$$

and the lines on a point P by

$$L(P) = \{L \in L \mid P \; I \; L\} \; .$$

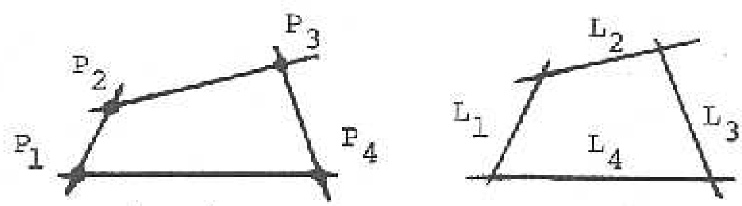
A concrete plane (as opposed to an abstract one) is one in which each line $L = P(L)$ <u>is</u> the set of its points and the incidence relation is just membership, $P \; I \; L \iff P \in L$. For each plane

$\Pi = (\mathcal{P}, \mathcal{L}, I)$ we have a <u>concrete</u> <u>realization</u> $\Pi_c = (\mathcal{P}, \mathcal{L}_c, \in)$ with the same points as Π, whose lines are all "concrete lines" $L_c = \mathcal{P}(L)$ for $L \in \mathcal{L}$, and whose incidence relative $P\ I_c\ L_c$ is inclusion $P \in \mathcal{P}(L)$. The map τ sending $P \to P$, $L \to \mathcal{P}(L)$ is a homomorphism of Π onto its concrete realization Π_c. Clearly τ is bijective on points, surjective on lines, and preserves incidence: $P\ I_c\ L_c \Leftrightarrow P \in \mathcal{P}(L) \Leftrightarrow P\ I\ L$. Therefore τ is an isomorphism iff it is injective on lines.

In general, we say a plane is <u>concrete</u> if the homomorphism $\Pi \xrightarrow{\tau} \Pi_c$ is an <u>isomorphism</u>; this is equivalent to saying that a line L is uniquely determined by its points $\mathcal{P}(L)$, $\mathcal{P}(L) = \mathcal{P}(L') \Rightarrow L = L'$. All the planes we shall be interested in will be concrete in this sense, so we can (and should) <u>think</u> of lines as being certain collections of points, but we will be very careful not to <u>write</u> $P \in L$ (but rather $P \in \mathcal{P}(L)$).

One reason for keeping to the abstract view is to maintain symmetry between points and lines. The <u>dual</u> of a plane $\Pi = (\mathcal{P}, \mathcal{L}, I)$ is $\Pi^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ where the points $\mathcal{P}^* = \mathcal{L}$ of the dual are the lines of the original, the lines $\mathcal{L}^* = \mathcal{P}$ of the dual are the original points, and the incidence relation $I^* = I^{-1}$ is the converse relation $L\ I^*\ P \Leftrightarrow P\ I\ L$ (ie $(L,P) \in I^* \Leftrightarrow (P,L) \in I$). <u>Principle</u> <u>of</u> <u>Duality</u> says that if a statement S phrased in terms only of points, lines, and incidence is true for all planes, then the dual statement S^*, obtained by interchanging "point" and "line" in statement S , is also true of all planes. The reason is that statement S^* for an arbitrary plane Π is really just statement S for Π^* . Note that the dual of a concrete plane is no longer concrete, so if we defined lines to be certain collections of points we could not avail ourselves of duality.

To weed out some degenerate examples we assume that a plane has "enough" points and lines. A <u>4-point</u> consists of 4 distinct points $\{P_1, P_2, P_3, P_4\}$ no 3 of which are collinear. A <u>4-line</u> consists of 4 lines $\{L_1, L_2, L_3, L_4\}$ no 3 of which are concurrent.



A <u>projective plane</u> is a plane which satisfies the axioms

(1.1) (Proj I) Any two distinct points lie on a unique line
(Proj II) Any two distinct lines lie on a unique point
(Proj III) There exists a 4-point.

A plane satisfying Proj I and II but not Proj III is called a <u>degenerate</u> projective plane; there are only 4 possible kinds of degenerate planes (see Problem Set 1).

We obtain a <u>category of projective planes</u> by taking as objects all projective planes and as morphisms all isomorphisms. (Projective planes are very rigid objects, like division rings, so that the only homomorphisms which are not isomorphisms are degenerate; see Problem Set #4).

It is important that duality applies to projective planes. Clearly Proj I and II are dual axioms; to see that the dual plane Π^* satisfies Proj III we must show Π satisfies the dual axiom

(Proj III*) There exist a 4-line.

This follows from

1.2 (Lemma) If $\{P_1,P_2,P_3,P_4\}$ is a 4-point and L_{ij} denotes the unique line on P_i and P_j, then $\{L_{12},L_{23},L_{34},L_{41}\}$ is a 4-line.

Proof. The P_i are distinct, so by Proj I there exists a unique line L_{ij} on P_i and P_j for $i \neq j$. Suppose three of the sides L_{ii+1} are concurrent; by cyclic symmetry we may suppose L_{12},L_{23},L_{34} all lie on a point P. Now either $P \neq P_2$ or $P \neq P_3$, say $P \neq P_2$. Then L_{12} and L_{23} have 2 distinct points P and P_2 in common, so by Proj I $L_{12} = L_{23}$. But then P_1,P_2,P_3 all lie on $L_{12} = L_{23}$, contrary to non collinearity. (If $P \neq P_3$ we would have P_2,P_3,P_4 collinear). ∎

This establishes

1.3 (Duality Theorem) The dual of a projective plane is projective. ∎

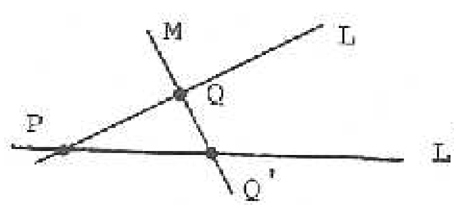
1.4 (Duality Principle) If S is a statement true of all projective planes, so is the dual statement S^* obtained by everywhere interchanging "points" and "lines." ∎

The virtue of duality is that you get two theorems for the proof of one.

Next we see that projective planes are concrete.

1.5 (Concreteness Theorem) Projective planes are concrete.

Proof. We must show distinct lines consist of different points, $L \neq L' \Rightarrow P(L) \neq P(L')$. Now if $L \neq L'$ then by Proj II L, L' intersect at a unique point P. By Proj III* there exists 4 lines no 3 of which are concurrent; in particular, these lines don't all pass through P, say M is off P (so $M \neq L, L'$). By Proj II M intersects L,L at points Q, Q' which are on M and therefore different from P; by Proj I L is the unique line through P and Q, L' the unique line through P and Q'. Since $L \neq L'$ we have $Q \neq Q'$, so $Q \in P(L)$ but $Q \notin P(L')$ (else $Q \in P(L') \cap P(M) = \{Q'\}$). ∎



This justifies always thinking of lines as collections of points.

We write $P \vee Q$ for the unique line on $P \neq Q$ (the <u>span</u> of P and Q) guaranteed axiom Proj I, and $L \wedge M$ for the unique point incident to L and M (the point of <u>intersection</u>) guaranteed by Proj II.

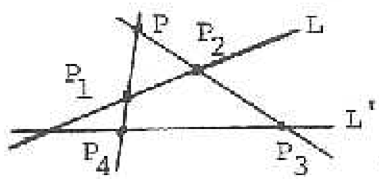
WARNING: One must always check that $P \neq Q$ in any specific instance before writing $P \vee Q$; $P \vee P$ is meaningless. The same goes for $L \wedge M$.

One of the major goals of projective geometry is to classify all projective planes up to isomorphism. In particular, one would like to classify all finite projective planes. We are still a long way from this goal. In fact, a major open question is the cruder classification of finite projective planes according to cardinality: what are the possible cardinalities for a finite projective plane?

There is an important numerical invariant of a projective plane, namely the number of points on a line. Before counting the number of points on a line we need to be able to move off lines.

1.6 Lemma. Given any two lines L, L' there exists a point P off both of them.

Proof. Suppose on the contrary all points lie on L or L′. Given a quadrangle as in Proj. III we can't have 3 points on either line, so two must be on L and two on L′: relabel so P_1, P_2 are on L and P_3, P_4 on L′. We claim $P = (P_1 \vee P_4) \wedge (P_2 \vee P_3)$ lies on neither. Notice $\wedge$ exists since



$P_1 \vee P_4 \neq P_2 \vee P_3$, indeed $L \wedge (P_1 \vee P_4) = P_1 \neq P_2 = L \wedge (P_2 \vee P_3)$ (the latter $\wedge$'s exist since $L \neq P_1 \vee P_4$ as P_4 is off L, similarly $L \neq P_2 \vee P_3$). Then P is on $P_1 \vee P_4$ and $P_2 \vee P_3$, so if it were on L too we would have $L \wedge (P_1 \vee P_4) = P = L \wedge (P_2 \vee P_3)$, contradiction. Similarly P is off L'. ∎

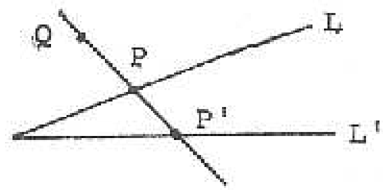
1.7 (Cardinality Theorem) Any two lines L,L' in a projective plane have the same number of points: $|P(L)| = |P(L')|$. In fact, if Q is off L and L' then the map

$$\pi^{Q}_{L,L'}(P) = L' \wedge (Q \vee P)$$

defines a bijection of L on L'.

Proof. By the preceeding Lemma, there exists a Q off L and L'. Once we have such Q, for any P on L we have $Q \neq P$

(Q is not on L) so $Q \vee P$ exists, and $L' \neq Q \vee P$ since Q isn't on L', so $L' \wedge (Q \vee P) = P'$ exists. The map $P \to P'$ is a bijection $\pi_{L,L'}$ since it has inverse $\pi_{L',L}$: $\pi_{L',L}(\pi_{L,L'}(P)) = L \wedge (Q \vee P')$ $= L \wedge (Q \vee P) = P$ ($Q \vee P$ contains Q and $P' = L' \wedge (Q \vee P)$,



as does $Q \vee P'$, so by Proj I $Q \vee P = Q \vee P'$). Similarly $\pi_{L',L}\,\pi_{L,L'} = 1_{L'}$, so $\pi_{L',L}$ and $\pi_{L,L'}$ are inverse bijections. ∎

The number of points on a line is thus an invariant of the plane. A projective plane has <u>order n</u> if every line has n+1 points. (Warning: the order is <u>not</u> the cardinality n in the projective case; the definition of order will become clearer in the affine case). It is an open question what n can be the order of a projective plane; so far all known planes have order of the form $n = p^k$ (and there always exist planes of order p^k).

1.8 (Cardinality Conjecture) Every finite projective plane has prime power order $n = p^k$. ○

The most celebrated positive result is the

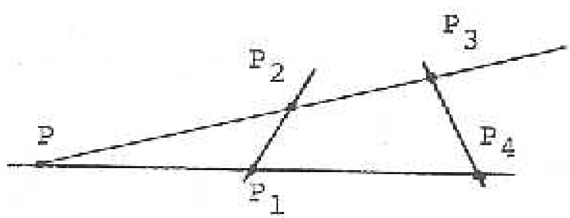
1.9 (Bruck-Ryser Theorem) If $n \equiv 1$ or $n \equiv 2 \bmod 4$ and if $n \neq a^2 + b^2$ cannot be written as a sum of squares of integers a,b then n cannot be the order of a projective plane. □

For example, this rules out 6, 14, 21, 22,... But it leaves the OPEN QUESTION of the existence of a plane of order 10. (Computers are little help; there are $2^{10,000}$ possibilities for an incidence matrix of order 10).

The Cardinality Theorem has some useful consequences

1.10 (Cardinality Corollary) All lines contain at least 3 points: $|P(L)| \geq 3$.

Proof. It suffices to find one line with ≥ 3 points. If $\{P_i\}$ form a 4-point then $P_1 \vee P_4$ contains 3 distinct points $P_1, P_4, (P_1 \vee P_4) \wedge (P_2 \vee P_3) = P$: $P \neq P_1$ since P_1 is not on $P_2 \vee P_3$ (P_1, P_2, P_3 are not collinear), similarly $P \neq P_4$. ■



Using this can actually construct 4-points all over the place.

1.11 Example: (Vector-space planes). Let V be any 3-dimensional left vector space over an (associative) division ring Δ. Set

$$\mathrm{Proj}(V) = (P(V), L(V), I(V))$$

$$P(V) = \{\text{1-dimensional subspaces of } V\}$$

$$L(V) = \{\text{2-dimensional subspaces of } V\}$$

$$I(V) = \text{inclusion } (P \; I \; L \iff P \subset L)$$

We claim Proj(V) is a projective plane, the <u>plane of the vector space V</u> or a <u>vector-space plane</u>.

(Proj. I): If $P \neq P'$ are 1-dimensional subspaces then

$L = P + P'$ is 2-dimensional, hence a line through P and P'; it is unique since if L' passes through P and P' it must contain P, P', and $P + P'$, and if it is to be 2-dimensional it can't contain anything else.

(Proj. II) If $L \neq L'$ are 2-dimensional then $L + L' > L$ implies $L + L' = V$ is 3 dimensional, so from the dimension formula $\dim L \cap L' = \dim L + \dim L' - \dim(L + L') = 2 + 2 - 3 = 1$ we see $L \cap L' = P$ is a point on L and L'. Any other P' on L and L' has $P' \subset L \cap L' = P$, and if P' is also to have dimension 1 we must have $P' = P$.

(Proj. III) If v_1, v_2, v_3 are a bases for V over Δ we claim $P_1 = \Delta v_1$, $P_2 = \Delta v_2$, $P_3 = \Delta v_3$, $P_4 = \Delta(v_1 + v_2 + v_3)$ form a 4-point: no 3 of the points are collinear since any 3 of v_1, v_2, v_3, $v_1 + v_2 + v_3$ span V.

We claim any semilinear isomorphism $V \xrightarrow{T} \tilde{V}$ of vector spaces over $\Delta, \tilde{\Delta}$ (ie T is additive and $T(\delta v) = \tau(\delta)T(v)$ for some isomorphism $\Delta \xrightarrow{\tau} \tilde{\Delta}$) induces an isomorphism $\mathrm{Proj}(V) \xrightarrow{\mathrm{Proj}(T)} \mathrm{Proj}(\tilde{V})$ of planes. Certainly a semi-linear T induces a bijection from 1-dimensional Δ-spaces to 1-dimensional $\tilde{\Delta}$-spaces, and similarly on 2-dimensional spaces. It preserves incidence since it preserves inclusion. This gives us a functor

$$\text{Vector spaces} \xrightarrow{\mathrm{Proj}} \text{Projective planes}$$

from the category of 3-dimensional vector spaces (V,Δ) over division rings, with semilinear isomorphisms (T,τ) as morphisms, to the category of projective planes by $V \to \mathrm{Proj}(V)$, $T \to \mathrm{Proj}(T)$.

The Fundamental Theorem of Projective Geometry says all Desarguian planes are vector-space planes and (almost) all isomorphisms come from semilinear isomorphisms of vector spaces. ■

1.12 Example: (Division-ring planes). If we are given only a division ring Δ we can form a canonical 3-dimensional left vector space $V(\Delta) = \Delta^3$ over Δ. The projective plane $\mathrm{Proj}(V(\Delta))$ we denote simply by $\mathrm{Proj}(\Delta)$; such a plane is called a division-ring plane. (If $\Delta = \Phi$ is a field it is called a field plane). Any ring isomorphism $\Delta \xrightarrow{\tau} \Delta'$ induces a semilinear isomorphism $\Delta^3 \xrightarrow{T} \Delta'^3$ by $T(\delta_1,\delta_2,\delta_3) = (\tau(\delta_1),\tau(\delta_2),\tau(\delta_3))$; we denote this by $\mathrm{Proj}(\tau)$. The correspondence $\Delta \to \mathrm{Proj}(\Delta)$, $\tau \to \mathrm{Proj}(\tau)$ determines a functor

$$\text{Division rings} \xrightarrow{\mathrm{Proj}} \text{Projective planes}$$

from the category of division rings and isomorphisms to the category of projective planes. ■

1.13 Example: (Field Planes). There is an alternate description of field planes which is more standard in the study of projective geometry. We define two 3-tuples $(\alpha_1,\alpha_2,\alpha_3)$ and $(\alpha_1',\alpha_2',\alpha_3')$ to be equivalent if they differ by a nonzero scalar; the equivalence classes $[(\alpha_1,\alpha_2,\alpha_3)]$ of nonzero vectors are just the one-dimensional subspaces $\Phi(\alpha_1,\alpha_2,\alpha_3)$ (minus zero). For projective 3-space over Φ we take $P = L$ to be these equivalence classes. This is reasonable: a 2-dimensional subspace L is uniquely determined by its orthogonal complement relative to the canonical inner product $<(\alpha_1,\alpha_2,\alpha_3),(\beta_1,\beta_2,\beta_3)> = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$ on Φ^3 ; this complement has the form $L^{\perp} = \Phi(\beta_1,\beta_2,\beta_3)$ for some nonzero $(\beta_1,\beta_2,\beta_3)$, so we identity L with $[(\beta_1,\beta_2,\beta_3)]$. Now the incidence relation P I L becomes $P \in L = L^{\perp\perp}$ or $P \perp L^{\perp}$, so $[(\alpha_1,\alpha_2,\alpha_3)]I[(\beta_1,\beta_2,\beta_3)] \Longleftrightarrow \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$, or

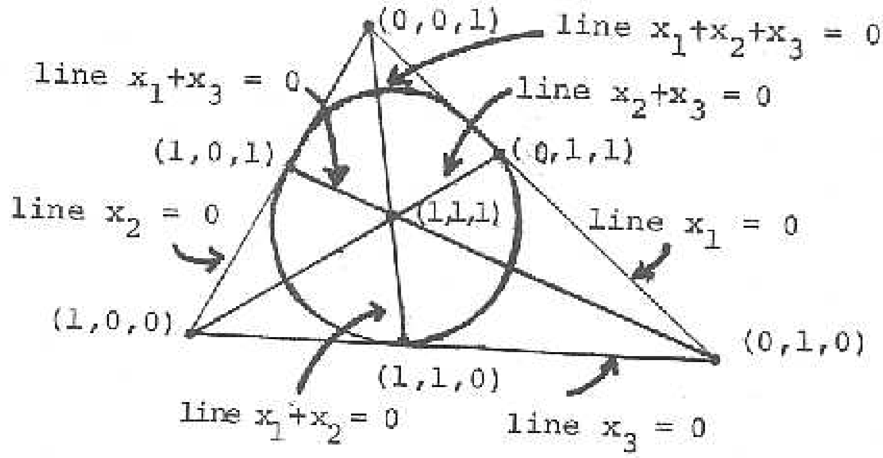
$$P \text{ I } L \Longleftrightarrow <P,L> = 0.$$

An invertible semilinear transformation $\Phi^3 \xrightarrow{T} \Phi'^3$ induces an isomorphism $\ulcorner T \urcorner$ by

$$\ulcorner T \urcorner (P) = T(P)$$
$$\ulcorner T \urcorner (L) = T^{*-1}(L).$$

The reason for the inverse adjoint is that $<\ulcorner T \urcorner(P), \ulcorner T \urcorner(L)> = <T(P), T^{*-1}(L)> = <P, T^{*}T^{*-1}(L)>$ (by definition of adjoint) $= <P,L>$ yields $\ulcorner T \urcorner(P) I \ulcorner T \urcorner(L) \Longleftrightarrow P \text{ I } L$.

As a particular case, the world's smallest field $\Phi = \mathbb{Z}_2$ gives rise to a projective plane $\text{Proj}(\Phi) = (\Phi^3 \setminus 0)/\Phi^*$ with $\dfrac{2^3 - 1}{2 - 1} = 7$ elements, the Fano plane



World's Smallest Projective Plane

Since this contains only 3 points on each line, it is as small as possible. Further, one can show (exercise!) this is the unique plane of order 2 (with 7 points).

1.14 Example: (Incidence matrices). A practical method of describing finite projective planes is by means of the incidence matrix $M(\Pi) = (\mu_{ij})$ where we have numbered the points P_i and lines L_j and set $\mu_{ij} = \begin{cases} 1 & \text{if } P_i \text{ I } L_j \\ 0 & \text{if } P_i \not\text{I } L_j \end{cases}$. The matrix $M(\Pi)$ has the properties

(MI) each row (and column) have n 1's

(MII) two rows (or columns) have exactly one 1 in a common place

(MIII) there exist 4 rows (or columns) no 3 of which have a 1 in a common place.

Any $n \times n$ matrix $M = (\mu_{ij})$ of 0's and 1's satisfying MI-III defines a projective plane $\Pi = \Pi(M)$ by letting $P = \{P_1, \ldots P_n\}$, $L = \{L_1, \ldots L_n\}$ with $P_i \ I \ L_j \iff \mu_{ij} = 1$. Indeed, the row version of MII is Proj I: given P_i, P_j there is a unique k such that $\mu_{ik} = \mu_{jk} = 1$, ie a unique L_k such that $P_i \ I \ L_k$, $P_j \ I \ L_k$. The row version of MII is Proj II. MIII is Proj III: there are 4 points P_1, P_2, P_3, P_4 no 3 of which are collinear (ie $\mu_{i\ell} = \mu_{j\ell} = \mu_{k\ell} = 1$ share a common 1 in the ℓ^{th} place, ie. lie on L_ℓ).

Here the incidence matrix of the dual plane is just the transpose $M(\Pi^*) = M(\Pi)^t$ of the original incidence relation.

An isomorphism $\Pi(M) \xrightarrow{\sigma} \Pi(M')$ corresponds to a pair of permutations π, ρ of $\{1, \ldots, n\}$ (where $\sigma(P_i) = P'_{\pi(i)}$, $\sigma(L_i) = L'_{\rho(i)}$) satisfying $\mu_{\pi(i)\rho(j)} = \mu_{ij}$. ■

Exercises

1.1 Show $\Pi^{**} \cong \Pi$ for every plane Π. Give an example of a projective plane Π where $\Pi \not\cong \Pi^*$.

1.2 Define imbedding $\Pi \xrightarrow{\sigma} \tilde{\Pi}$ of projective planes. Show the following are equivalent for an injective σ:

(i) σ is an imbedding

(ii) $\sigma(P \vee Q) = \sigma(P) \vee \sigma(Q) \quad (P \neq Q)$

(iii) $\sigma(L \wedge M) = \sigma(L) \wedge \sigma(M) \quad (L \neq M)$

(iv) P, Q, R collinear $\Rightarrow \sigma(P), \sigma(Q), \sigma(R)$ collinear

(v) L, M, N concurrent $\Rightarrow \sigma(L), \sigma(M), \sigma(N)$ concurrent.

Condition (iv) is the reason that an automorphism of Π is called a collineation - it preserves the relation of collinearity.

1.3 If $P(\Pi) \xrightarrow{\sigma} P(\tilde{\Pi})$ is a bijection preserving collinearity, show σ can be extended to an isomorphism of planes. Thus an isomorphism is completely determined by its action on points, the lines following automatically.

1.4 If $\{P_1, P_2, P_3, P_4\}$ is a 4-point and $L_{ij} = P_i \vee P_j$, show no 4 of the 7 points $P_1, P_2, P_3, P_4, P_5 = L_{12} \wedge L_{34}$, $P_6 = L_{23} \wedge L_{41}$, $P_7 = L_{24} \wedge L_{31}$ are collinear. Deduce directly from this that if Π is a projective plane and L, L' any two lines in Π then there exists a point P off L and L'.

1.6 If σ is an involution (= automorphism of period 2, $\sigma^2 = 1$) of a finite projective plane Π show each point lies on at least one fixed line $\sigma(L) = L$ and each line contains at least one fixed point $\sigma(P) = P$.

1.7 If P,L are two sets and $\vee, \wedge$ are symmetric maps $P \vee P \to L$, $L \wedge L \to P$ defined for $P \neq Q$ or $L \neq M$ satisfying

$$\text{(i)} \quad P \vee Q \neq P \vee Q' \Longrightarrow (P \vee Q) \wedge (P \vee Q') = P$$

$$\text{(i)}^* \quad L \wedge M \neq L \wedge M' \Longrightarrow (L \wedge M) \vee (L \wedge M') = L$$

show the incidence relation $P \; I \; L \Longleftrightarrow L = P \vee Q$ for some Q (dually $\Longleftrightarrow P = L \wedge M$ for some M) defines a plane $\Pi = (P,L,I)$ satisfying Proj I,II. Show Π is projective iff there are distinct P_1, P_2, P_3 such that $P_1 \vee P_2$, $P_2 \vee P_3$, $P_3 \vee P_1$ are distinct. Dualize.

1.8 Take P = {1-dimensional subspaces}, L = {2-dimensional subspaces} in a 3-dimensional left vector space V over Δ, set $P \vee Q = P + Q$, $L \wedge M = L \cap M$ for $P \neq Q$, $L \neq M$. Show these satisfy the conditions of #7. Show the resulting projective plane is just Proj(V).

Problem Set on Cardinality

1. Show $|P(L)| = |P(L')|$ for any two lines in a projective plane by defining projections $\pi: P(L) \to P(L')$, $\pi': P(L') \to P(L)$ with $\pi \circ \pi' = 1_{L'}$, $\pi' \circ \pi = 1_L$.
2. Show $|P(L)| = |L(P)|$ for any point P off L by defining $f: P(L) \to L(P)$, $g: L(P) \to P(L)$ with $f \circ g = id_{L(P)}$, $g \circ f = id_{P(L)}$. Conclude from this (rather than #1) that $|P(L)| = |P(L')|$ for $L \neq L'$. Thus each line is on N points and each point on N lines.
3. If one line has $|P(L_\circ)| = N$ points show there are $|P(\Pi)| = N^2 - N + 1$ points (and lines) in Π.

Problem Set on Subplanes

1. $\Pi' = (P',L',I')$ is a <u>subplane</u> of $\Pi = (P,L,I)$ if $P' \subset P$, $L' \subsetneq L$, $I' = (P' \times L') \cap I$, (or Π is an <u>extension</u> of Π') and (i) $P',Q' \in P' \Rightarrow P' \vee Q' \in L'$, (i*) $L', M' \in L' \Rightarrow L' \wedge M' \in P'$ (ie P', L' closed under $\wedge, \vee$). Show Π' satisfies Proj I,II if Π does. Note that Proj III need not be inherited by Π'.
2. If $\phi: \Pi \longrightarrow \tilde{\Pi}$ is an imbedding, show $\phi(\Pi)$ is a subplane. If $\tilde{\Pi}'$ is a subplane of $\tilde{\Pi}$, show $\Pi' = \phi^{-1}(\tilde{\Pi}')$ is a subplane of Π.
3. If $\phi: \Pi \rightarrow \Pi$ is an automorphism, show the fixed points and lines $\phi(P) = P$, $\phi(L) = L$ form a subplane Π' of Π.