Appendix IV

Left Alternative Algebras

§1 Left Moufang algebras

A <u>left alternative</u> algebra is one satisfying the left alternative law

(1.1) $$x(xy) = x^2y. \qquad \text{(Left alternative)}$$

It turns out that this law alone is not enough to obtain satisfactory results in general, and we will concern ourselves entirely with <u>left Moufang</u> algebras - those which satisfy

(1.2) $$\{x(yx)\}z = x\{y(xz)\} \qquad \text{(Left Moufang)}$$

<u>in addition to</u> (1.1). In terms of operators the axioms are

(1.1 op) $$L_{x^2} = L_x^2$$

(1.2 op) $$L_{U(x)y} = L_xL_yL_x \qquad (U_xy = x(yx))$$

while terms of associators

(1.1a) $$[x,x,y] = 0$$

(1.2a) $$x[y,x,z] + [x,yx,z] = 0 \qquad \text{(Left Bumping)}$$

(note $x\{(yx)z - y(xz)\} + \{x(yx)\}z - x\{(yx)z\} = \{x(yx)\}z - x\{y(xz)\}$).

In the presence of a unit element, axiom (1.2) implies axiom (1.1) by substituting $y = 1$. Furthermore, in the presence of a scalar $\frac{1}{2}$ axiom (1.1) implies axiom (1.2),

1.3 (Equivalence Theorem) If 2 is injective or surjective on A, then A is left alternative iff it is left Moufang.

Proof. Always left Moufang implies left alternative (left Moufang requires both (1.1) and (1.2)). Now assume A is left alternative. Then we have

(1.4) $$2L_xL_yL_x = 2L_{U(x)y}$$

since $x\circ(x\circ y) - x^2\circ y = x(xy) + x(yx) + (xy)x + (yx)x - x^2y - yx^2$ $= -[x,x,y] + x(yx) + [x,y,x] + x(yx) + [y,x,x] = 2x(yx) + [x,y,x]$ $+ [y,x,x] - [x,x,y] = 2U_xy$ (in the presence of (1.1a) and its linearization), similarly on End A $2L_x(L_yL_x) = L_x\circ(L_x\circ L_y) - L_x^2\circ L_y$, so $2L_{U(x)y} = L_{x\circ(x\circ y)} - L_{x^2\circ y} = L_x\circ(L_x\circ L_y) - L_{x^2}\circ L_y = 2L_xL_yL_x$ (using (1.1 op) and its linearization). [We are just saying $2U_x = V_x^2 - V_{x^2}$ can be built from squares and circles, so if $x \longrightarrow L_x$ preserves these it preserves $2U_x$] .

If 2 is injective we can cancel it from the relation $2L_xL_yL_x\ z = 2L_{U(x)y}\ z$ in (1.4) to get the left Moufang law $L_xL_yL_x\ z = L_{U(x)y}\ z$ for all x,y,z. If 2 is surjective, every $z \in A$ has the form $z = 2w$, so applying (1.4) to w gives $L_xL_yL_x\ z = 2L_xL_yL_x\ w = 2L_{U(x)y}\ w = L_{U(x)y}\ z$. ∎ .

1.5 Remark. It will be important to observe that a left alternative or left Moufang algebra is actually alternative if and only if it is flexible, $[x,y,x] = 0$, since then right alternativity $[y,x,x] = -[x,y,x] = 0$ follows by linearized (1.1a)

(1.1a') $$[x,y,z] + [y,x,z] = 0.$$ ∎

Since the defining axioms (1.1),(1.2) are quadratic they remain valid in all extensions, so the scalar extension A_Ω of a left alternative or Moufang algebra A remains left alternative or Moufang. Furthermore, if A is left alternative or

Moufang, so is the algebra $\hat{A} = \Phi 1 + A$ obtained by adjoining a unit to A: for (1.1a) we have $[\alpha 1 + x, \alpha 1 + x, \beta 1 + y]$ $= [x,x,y] = 0$ since 1 is in the nucleus of $\hat{A}$, and similarly for (1.2a) $(\alpha 1 + x)[\beta 1 + y, \alpha 1 + x, \gamma 1 + z]$ $+ [\alpha 1 + x, (\beta 1 + y)(\alpha 1 + x), \gamma 1 + z] = (\alpha 1 + x)[y,x,z]$ $(\alpha 1 + x)[y,x,z] + [x, \beta x + \alpha y + yx, z] = \alpha\{[y,x,z] + [x,y,z]\}$ $+ \beta[x,x,z] + x[y,x,z] + [x,yx,z] = 0$. Indeed, we require a left Moufang algebra to satisfy (1.1) in addition to the left Moufang law precisely so that the unital hull A will remain left Moufang.

The linear mapping $A \to \text{End}\,\hat{A}$ given by $x \to L_x$ is injective since $L_x = 0 \Rightarrow x = L_x\hat{1} = 0$ (the left regular representation on A itself need not be injective).

1.5 **Proposition.** A linear algebra A is left Moufang iff the left regular representation $x \to L_x$ of A in $(\text{End}\,A)^+$ is a homomorphism of quadratic algebras.

Proof. The map is always linear, so it will be a homomorphism iff it preserves the quadratic operations $x^2 = xx$ and $U_x y = x(yx)$: $L_{x^2} = L_x^2$ and $L_{U(x)y} = L_x L_y L_x$. ∎

Note that if A is unital then $L_1 = 1$ and the left regular representation is a monomorphism of unital quadratic algebras. Since any A is imbedded in a unital algebra $\hat{A}$, we have as

1.7 Corollary. If A is left Moufang then the left regular representation $A \to \mathrm{End}(\hat{A})^{+}$ is a monomorphism of quadratic algebras. In particular, any identities satisfied by the quadratic algebra $\mathrm{End}(\hat{A})^{+}$ are satisfied by A as quadratic algebra. ∎

This immediately leads to some useful identities

(1.8) $$U_{x(yx)} = U_x U_y U_x \quad \text{(Fundamental Formula)}$$

(1.9) $$V_{x,y} U_x = U_x V_{y,x} = U_{x,x(yx)}$$

where $V_x = L_x + R_x$, $V_{x,y} = L_x L_y + R_{yx}$ are defined by $V_x y = x \circ y = xy + yx$ and $V_{x,y} z = U_{x,z} y = x(yz) + z(yx)$.
In addition we have

(1.10) $$L_{x^n} = L_x^n, \quad U_{x^n} = U_x^n$$

which also could have been proven directly from the axioms.
This has as immediate consequence

1.11 (Power-Associativity Theorem) A left Moufang algebra is strictly power-associative,

$$x^n x^m = x^{n+m}$$

where $x^n = L_x^n 1$ (i.e. $= L_x^{n-1} x$).

Proof. $x^n x^m = L_{x^n} L_{x^m} 1 = L_x^n L_x^m = L_x^{n+m} 1 = x^{n+m}$ by (1.10), and this remains valid in all scalar extensions A_Ω since they all remain left Moufang. ∎

This allows us to talk about idempotent elements ($e^2 = e$) and nilpotent elements ($z^n = 0$) in the accustomed way.

Exercises

We can base the study of left Moufang algebras on the products x^2 and x^3 rather than x^2 and $U_x y$. Clearly $x^3 = U_x x$ can be defined in terms of U's, and conversely we can recover $U_x y$ from linearizing x^3.

1.1 Show that A is left Moufang iff it strictly satisfies

(1.1) $[x,x,z] = 0$

$(1.\hat{2})$ $[x^2,x,z] = 0,$

or in operator notation

(1.1 op) $L_{x^2} = L_x^2$

$(1.\hat{2}$ op) $L_{x^3} = L_x^3 \qquad (x^3 = x^2x = xx^2).$

1.2 Show that if 2 is injective or surjective then (1.1) implies $(1.\hat{2})$.

1.3 Linearize $(1.\hat{2})$ and (1.2) to show

$$(R_{xz} - R_z R_x)V_x = [R_z, L_{x^2}]$$

$$[R_z, L_x L_y] = R_{y(xz)} - R_z R_{yx}$$

1.4 Show L_x, U_x are idempotent (resp. nilpotent) if x is idempotent (resp. nilpotent).

§2 Units

An element u is a <u>unit</u> for A if $ux = xu = x$ for all $x \in A$. An element is a unit for A^+ if $U_u x = x$, $U_x u = x^2$ for all x.

2.1 Proposition. An element u is a unit for A iff it is a unit for A^+.

Proof. Clearly if $ux = xu = x$ for all x we will have $U_u x = u(xu) = ux = x$ and $U_x u = x(ux) = xx = x^2$. Conversely, if u is a unit for A^+ then $u^2 = U_u u = u$ so $xu = L_{U(u)x}\, u = L_u L_x L_u\, u = u\{xu^2\} = u\{xu\} = U_u x = x$ and therefore $ux = u(xu) = U_u x = x$ too. ∎

2.2 Proposition. A has a unit if either one of

(i) some U_x is surjective

(ii) some L_x, R_x are surjective.

Proof. Since $U_x = L_x R_x$, (ii) implies (i), so we show only that (i) implies the existence of a unit.

There must be a y such that $x = U_x y$, and $y = U_x z$ for some z. Then $L_x L_y L_x = L_{U(x)y} = L_x$ shows $L_x L_y = I$ on $L_x A = A$ (if $U_x = L_x R_x$ is surjective so is L_x), and then $L_y = L_x L_z L_x$ injective implies L_x injective too, so L_x and L_y are inverses. Set $u = yx$. This u is a left unit since $L_u = L_{yx} = L_{x\{y(yx)\}} = L_{x(y^2 x)} = L_x L_y L_y L_x = I$, and a right unit since for any $a = U_x b$ in A we have $au = x\{b(xu)\} = x\{bx\} = a$ by $xu = x(yx) = x$. ∎

§3 Inverses

Two elements x,y are <u>inverses</u> in A if $xy = yx = 1$; they are inverses in A^+ if $U_x y = x$, $U_x y^2 = 1$.

3.1 Theorem. The following are equivalent:

(i) x,y are inverses in A

(ii) x,y are inverses in A^+

(iii) L_x, L_y are inverses.

In this case the inverse $y = x^{-1}$ is unique, and

$$L_{x^{-1}} = R_x U_x^{-1} = L_x^{-1},\ R_{x^{-1}} = L_x U_x^{-1} = L_x R_x^{-1} L_x^{-1},\ U_{x^{-1}} = U_x^{-1}.$$

Proof. (i) ⟹ (ii): if $xy = yx = 1$ clearly $U_x y = x(yx) = x$ and $U_x y^2 = x\{y^2 x\} = x\{y(yx)\} = xy = 1$.

(ii) ⟹ (iii): since $x \to L_x$ is a monomorphism of A^+ in $(\mathrm{End}\ A)^+$, inverses x,y in A^+ go into inverses L_x, L_y in $(\mathrm{End}\ A)^+$, and $L_x L_y L_x = L_x, L_x L_y^2 L_x = I$ shows L_x is invertible with inverse L_y in End A.

(iii) ⟹ (i): $1 = L_x L_y 1 = xy$ and similarly $1 = yx$.

In this case $U_x U_y U_x = U_{U(x)y} = U_x$ and $U_x U_{y^2} U_x = U_{U(x)y^2} = I$ (by the Fundamental Formula (1.8)) imply U_x, U_y are inverses, $U_{x^{-1}} = U_x^{-1}$. We have $L_{x^{-1}} = L_x^{-1}$ by (iii), and $L_x R_x U_x^{-1} = I$ implies $R_x U_x^{-1} = L_x^{-1}$. Cancelling in $L_{x^{-1}} R_{x^{-1}} U_x = U_{x^{-1}} U_x = I$ gives $R_{x^{-1}} = L_x U_x^{-1}$; since L_x and $U_x = L_x R_x$ are invertible, so is R_x, thus $U_x^{-1} = R_x^{-1} L_x^{-1}$ and $R_{x^{-1}} = L_x R_x^{-1} L_x^{-1}$. ∎

Note that if A is not alternative we need <u>not</u> have $R_{x^{-1}} = R_x^{-1}$, although R_x is invertible when x is; indeed R_x^{-1} coincides with its conjugate $R_{x^{-1}} = L_x R_x^{-1} L_x^{-1}$ iff L_x and R_x commute.

The conditions on an element for it to be invertible are given by

3.2 (Inverse Theorem). The following are equivalent for an element x of a unital left Moufang algebra:

(i) x is invertible, $xy = yx = 1$ for some y

(ii) $xz = yx = 1$ for some y,z

(iii) $1 \in \text{Range } L_x \cap \text{Range } R_x$

(iv) $1 \in \text{Range } U_x$

(v) L_x, R_x are invertible

(vi) U_x is invertible.

Proof. Clearly (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii), and (ii) $\Rightarrow$ (i) since it implies $xy = x\{y(xz)\} = \{x(yx)\}z = xz = 1$. Clearly (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv), and we remarked (i) $\Rightarrow$ (v) after the previous theorem; (iv) $\Rightarrow$ (vi) since if $U_x z = 1$ then $U_x U_z U_x = I$ (via the Fundamental Formula (1.8)) shows U_x has left and right inverse; (vi) $\Rightarrow$ (i) since if $U_x y = x$ then $U_x U_y U_x = U_x$ implies $U_x U_y = I$, $U_x y^2 = U_x U_y 1 = 1$, x is invertible in A^+ and (by the previous theorem) also in A. ∎

Inverses can be used to characterize left Moufang algebras. A nonassociative division ring is said to have the <u>left inverse property</u> if

(3.3) $$x^{-1}(xy) = y \qquad (x \neq 0, \text{ all } y)$$

where x^{-1} denotes the left inverse of x, ie. the unique element satisfying $x^{-1}x = 1$. Then also $xx^{-1} = 1$, ie. x is the left inverse of its left inverse x^{-1}: $(x^{-1})^{-1} = x$. To see this, note $(x^{-1})^{-1} = (x^{-1})^{-1}\cdot 1 = (x^{-1})^{-1}\cdot x^{-1}x = x$ by the left inverse property applied to x^{-1}. More generally, for all y we have $x(x^{-1}y) = y$. In operator terms these mean

$L_{x^{-1}}L_x = L_xL_{x^{-1}} = I$ or

(3.4) $$L_{x^{-1}} = L_x^{-1} \qquad (x \neq 0).$$

3.5 (Left Inverse Property Theorem). A nonassociative division ring is left Moufang iff it has the left inverse property.

Proof. Our previous work on inverses has shown every left Moufang ring (=algebra over $\mathbb{Z}$) has the left inverse property. Conversely, assume A has the left inverse property. In Problem Set #2 we have already indicated one way to derive the left Moufang formula. What was tacitly behind that proof (due to Bruck) is the Hua formula

$$\{a^{-1} - (a + b^{-1})^{-1}\}^{-1} = a + aba$$

valid in any associative algebra where the inverses make sense (namely when a,b, $a + b^{-1}$ are invertible - then automatically $a^{-1} - (a + b^{-1})^{-1}$ is invertible with inverse $a + aba$: for example, $(a + aba)(a^{-1} - (a + b^{-1})^{-1}) = (a + b^{-1})ba \cdot a^{-1} - ab(a + b^{-1}) \cdot (a + b^{-1})^{-1} = (a + b^{-1})b - ab = 1$).

The basic idea is simple: Hua's formula says xyx can be build out of addition, subtraction, and inversion, so any map $x \to L_x$ preserving these operations will preserve xyx, giving left Moufangitivity $L_{xyx} = L_x L_y L_x$.

We must be a little careful in carrying out the details of this argument, for we do not know Hua applies to the left alternative algebra A (only to the associative algebra End(A)). We begin by noting

(3.6) $$L_x + L_x L_y L_x = L_{\{x^{-1} - (x+y^{-1})^{-1}\}^{-1}}$$

since $L_x + L_x L_y L_x = \{L_x^{-1} - (L_x + L_y^{-1})^{-1}\}^{-1}$ (associative Hua in End A) $= \{L_{x^{-1}} - L^{-1}_{x+y^{-1}}\}^{-1} = \{L_{x^{-1} - (x+y^{-1})^{-1}}\}^{-1}$ $= L_{\{x^{-1} - (x+y^{-1})^{-1}\}^{-1}}$ (by left inverse property in A). We don't yet know $\{x^{-1} - (x + y^{-1})^{-1}\}^{-1}$ equals $x + xyx$ in A since we don't know Hua for A , but when we apply (3.6) to the element 1 we get a nonassociative Hua formula $x + x(yx) = \{x^{-1} - (x + y^{-1})^{-1}\}^{-1}$. Thus (3.6) becomes $L_x + L_x L_y L_x = L_x + L_{x(yx)}$, so $L_x L_y L_x = L_{x(yx)}$ and A is left Moufang.

As usual, we have glibly passed over the cases $x = 0, y = 0$, $x + y^{-1} = 0$ when the requisite inverses don't exist. But $L_x L_y L_x = L_{x(yx)}$ is trivial if x or y is 0, and if $-x = y^{-1}$ then $y = -x^{-1}$ and $x(yx) = -x$, so $L_x L_y L_x = -L_x L_x^{-1} L_x = -L_x$. ∎

Since dually right Moufang is equivalent to right inverse property, and left + right Moufang is equivalent to alternativity, we have the characterization of alternative algebras in terms of inverses which we promised in Section I.4:

3.7 (Inverse Property Theorem) A nonassociative division ring is alternative iff it has the inverse property (ie. both left and right inverse properties). ∎

Now we turn to showing that for a division algebra, left Moufang alone already implies alternativity (thus the left inverse property implies alternativity).

§4 Skornyakov's Theorem

As in the case of alternative division algebras, we will be able to classify all left Moufang division algebras. Since we already know that any alternative division algebra is either associative or a Cayley algebra, all we need do is show a left Moufang division algebra is actually alternative (that is, we must establish flexibility). In the next section we will show that left alternativity is not sufficient - there exist left alternative division rings which are not left Moufang (much less alternative); of course, this can happen only in characteristic 2.

Let us introduce two operators

(4.1) $A_{x,y} = L_{xy} - L_x L_y \qquad B_{x,y} = L_{xy} - L_y L_x$.

Clearly left alternativity gives

(4.2) $A_{x,x} = B_{x,x} = 0$

(4.2') $A_{x,y} + A_{y,x} = B_{x,y} + B_{y,x} = 0$.

In the presence of the left Moufang Law we will establish

(4.3) $B_{x,y} L_x A_{x,y} = 0$

(4.3') $B_{x,y} A_{x,y} = 0$

(4.4) $A_{x,y} L_x B_{x,y} = L_{A(x,y)\ x[x,y]}$

(4.4') $A_{x,y} B_{x,y} = L_{A(x,y)[x,y]}$

(4.5) $B_{x,y}[x,y,x] = 0$

(4.6) $L_{[x,y,x]} = A_{xy,x} + A_{x,y} L_x = B_{xy,x} + L_x B_{x,y}$.

Observe that (4.3') and (4.4') follow from (4.3) and (4.4) in the left Moufang algebra $\hat{A}$ by linearizing $x \to x,1$ (since

$A_{x,y}, B_{x,y}$, and $[x,y]$ all vanish when $x = 1$). Therefore we must check only (4.3), (4.4), (4.5), (4.6).

For (4.3) we compute $B_{x,y} L_x A_{x,y} = \{L_{xy} - L_y L_x\} L_x \{L_{xy} - L_x L_y\} = L_{xy} L_x L_{xy} - L_y L_x^2 L_{xy} - L_{xy} L_x^2 L_y + L_y L_x^3 L_y = L_{(xy)\{x(xy)\}} - \{L_y L_{x^2} L_{xy} + L_{xy} L_{x^2} L_y\} + L_y L_{x^3} L_y = L_{(xy)(x^2y)} - \{L_{y(x^2(xy))} + L_{(xy)(x^2y)}\} + L_{y(x^3y)} = 0$ by repeated use of left Moufangitivity (recall $x^2(xy) = x^3y$ by (1.10)).

(4.4) is similar but messier: $A_{x,y} L_x B_{x,y} = \{L_{xy} - L_x L_y\} L_x \{L_{xy} - L_y L_x\} = L_{xy} L_x L_{xy} - L_x L_y L_x L_{xy} - L_{xy} L_x L_y L_x + L_x L_y L_x L_y L_x = L_{(xy)\{x(xy)\}} - L_{x(yx)} L_{xy} - L_{xy} L_{x(yx)} + L_x L_{y(xy)} L_x = L_{(xy)(x^2y)} - L_{\{x(yx)\}(xy)} - L_{(xy)\{x(yx)\}} + L_{x\{y(xy)\cdot x\}} = L(\{xy\}\{x(xy)\} - x\{y\{x(xy)\}\} - (xy)\{x(yx)\} + x\{y\{x(yx)\}\}) = L(\{xy\}\ x[x,y] + x\{y(x[y,x])\}) = L(A_{x,y}\ x[x,y])$.

(4.5) follows by applying (4.3)' to x ($[x,y,x] = A_{x,y}\ x$). For (4.6) note $L_{[x,y,x]} = L_{(xy)x} - L_{x(yx)} = L_{(xy)x} - L_x L_y L_x$ by left Moufang, while $A_{xy,x} + A_{x,y} L_x = \{L_{(xy)x} - L_{xy} L_x\} + \{L_{xy} - L_x L_y\} L_x = L_{(xy)x} - L_x L_y L_x = \{L_{(xy)x} - L_x L_{xy}\} + L_x \{L_{xy} - L_y L_x\} = B_{xy,x} + L_x B_{x,y}$ straight from the definitions (4.1). ∎

Now let us consider left Moufang algebras with the following property K (as in Kleinfeld):

(K) If z has the form (i) x, (ii) $[x,y]$, (iii) $x[x,y]$, or (iv) $x^2[x,y]$ then $[x,y,z]^2 = 0$ implies $[x,y,z] = 0$.

Clearly (K) holds if A is a division algebra, or has no zero divisors, or has no nilpotent elements, or even if it has no nilpotent associators $[x,y,p(x,y)]$ involving only 2 variables x,y (which associators are therefore zero in all alternative algebras).

Observe that if we multiply (4.4) or (4.4') on the right by $A_{x,y}$ the left sides vanish since both end in $B_{x,y}$ and by (4.3') $B_{x,y} A_{x,y} = 0$, therefore the right sides are zero and $L_{[x,y,z]} A_{x,y} = 0$ for $z = x[x,y]$ or $z = [x,y]$. If we apply this operator to z and recall $A_{x,y} z = [x,y,z]$ we see $[x,y,z]^2 = 0$ for $z = x[x,y]$ or $z = [x,y]$. Consequently, if we assume (K iii) or (K ii) we can conclude $[x,y,z] = 0$ for $z = x[x,y]$ or $[x,y]$. Putting $A_{x,y} x[x,y] = A_{x,y} [x,y] = 0$ back into (4.4), (4.4') shows that when A satisfies (K) we can strengthen them to read

(4.4K) $$A_{x,y} L_x B_{x,y} = 0$$

(4.4'K) $$A_{x,y} B_{x,y} = 0 .$$

This symmetry between A's and B's allows us to prove

4.7 (Property (K) Theorem). A left Moufang algebra is alternative iff it has property (K).

Proof. If A is alternative it has (K) because any associator $[x,y,z]$ vanishes when z is a polynomial in x and y (by Artin's Theorem). Conversely, suppose A has (K). We must show that A is flexible, i.e. all associators a = [x,y,x] vanish. Firstly,

(4.8) $$L_a^2 = A_{x,y} L_x^2 B_{x,y}$$

since $L_a^2 = \{-A_{x,y'} + A_{x,y} L_x\}\{-B_{x,y'} + L_x B_{x,y}\}$ (by (4.6) and (4.2'), y' = xy) where $A_{x,y'} B_{x,y'} = 0$ by (4.4'K) and $A_{x,y'} L_x B_{x,y} + A_{x,y} L_x B_{x,y'} = 0$ by linearized (4.4K). In particular, $a^3 = L_a^2 a = 0$ since $B_{x,y} a = 0$ by (4.3'). Then $(a^2)^2 = aa^3 = 0$ too ; but $a^2 = L_a^2 1 = A_{x,y} L_x^2 B_{x,y} 1 = [x,y,z]$ by (4.8) for $z = L_x^2 B_{x,y} 1 = x^2[x,y]$, so by (K iv) $(a^2)^2 = 0$ forces $a^2 = 0$, and since a = [x,y,z] for z = x we can apply (K i) again to get a = 0. Thus [x,y,x] = 0 , and any algebra with property (K) is flexible. ∎

Since all division algebras have (K), we have as an immediate consequence.

4.8 (Skornyakov's Theorem). A left Moufang division algebra is alternative, therefore associative or a Cayley division algebra. ∎

Since a division ring with left inverse property is left Moufang by 3.5, we have the geometrically significant

4.9 (Left Inverse Theorem) A nonassociative division ring with left inverse property is alternative, therefore associative or a Cayley algebra. ∎

As mentioned previously, a projective plane with enough translations is coordinatized by a division ring with left inverse property.

Problem Set on Jordan Homomorphisms

Let $F: A \to D$ be a Jordan homomorphism of the left Moufang algebra A into the associative algebra D, in the sense that F is a linear map satisfying

$$F(x^2) = F(x)^2 \qquad F(x(yx)) = F(x)F(y)F(x)$$

(for example, $D = \mathrm{End}\, A$ and $F(x) = L_x$).

Introduce the abbreviations

$$x^y = F(xy) - F(x)F(y) \qquad x_y = F(xy) - F(y)F(x) .$$

Thus F is a homomorphism iff all $x^y = 0$, and an antihomomorphism iff all $x_y = 0$.

1. Prove $x^x = x_x = 0$; linearize. Show $x^y - y_x = F([xy])$, $x^y - x_y = [F(y),F(x)]$.
2. Show $x_y x^y = 0$, $x^y x_y = F([x,y,[x,y]])$.
3. Show $x^{yx} = F(x)x^y$, $x_{yx} = x_y F(x)$.
4. Show $x_y F(x) x^y = 0$ (when $1/2 \in \Phi$ derive this immediately from #2, #3).
5. Show $F([y,x,x]) = x^{xy} - x^y F(x) = x_{xy} - F(x)x_y$.
6. Show $x \circ y = 0 \Longrightarrow F([y,x,x])x^y = 0$.
7. Show $x^y F(z) + F(z)x_y = F([x,y,z] + z[x,y])$, $x_y F(z) + F(z)x^y = F(-[x,y,z] + [x,y]z)$.
8. Show $x_y F([x,y,z]) + F([x,y,z])x^y = 0$.
9. Show $x^y F(z)x_y + x_y F(z)x^y = F([x,y,z[x,y]])$.
10. Show $x^y F(x)x_y = F([x,y,x[x,y]])$ and $F([x,y,x[x,y]])x^y = 0$.
11. When $F(x) = L_x$ show $x^y(z) = [x,y,z]$, $x_y(z) = [x,y]z + [x,y,z]$, $x_y(1) = [x,y]$. Simplify notation in #2, #7, #8, #9, #10.

§5 Bruck's examples

We know left alternative and left Moufang are the same in characteristic $\neq 2$. In this section we construct examples of left alternative division rings of characteristic 2 which are not left Moufang, much less alternative.

We begin with any unital commutative associative Φ-algebra Ω of characteristic 2, and any Φ-linear mapping s of Ω into itself. We define an algebra

(5.1) $$A(\Omega,\sigma) = \Omega 1 \oplus \Omega u$$

$$xy = (\alpha 1 \oplus \beta u)(\gamma 1 \oplus \delta u) = (\alpha\gamma + s(\beta)\ \delta)1 \oplus (\alpha\delta + \beta\gamma)u\ .$$

This is like the Cayley-Dickson formula, except we use $\beta\gamma$ instead of the expected $\beta s(\gamma)$. For fixed α,β this expression is linear in γ and δ; so L_x is Ω-linear (though R_y is only Φ-linear, due to the presence of the $s(\beta)$).

5.2 **Lemma.** $A(\Omega,\sigma)$ is always left alternative, but is left Moufang iff $s(q(x)\omega) = q(x)s(\omega)$ for all $\omega,\alpha,\beta \in \Omega$ where $q(x) = \alpha^2 + s(\beta)$. If Ω contains no nilpotent elements, the left Moufang condition is that $s = L_\sigma$ for $\sigma = s(1)$.

Proof. To check left alternativity $L_{x^2} = L_x^2$, note that for $x = \alpha 1 + \beta u$ we have $x^2 = (\alpha^2 + s(\beta)\beta)1 + (2\alpha\beta)u = q(x)1$, therefore $L_{x^2} = q(x)I$ since $L_{\omega 1} = \omega I$ (BEWARE: $L_{\omega u} \neq \omega L_u$ since R_y is not Ω-linear !), and $L_x^2 = (\alpha I + L_{\beta u})^2 = \alpha^2 I + L_{\beta u}^2 = q(x)I$ since we are in characteristic 2. (Note

$L_{\beta u} \cong \begin{pmatrix} 0 & s(\beta) \\ \beta & 0 \end{pmatrix}$, $L_{\beta u}^2 = \begin{pmatrix} s(\beta)\beta & 0 \\ 0 & \beta s(\beta) \end{pmatrix}$ relative to the obvious Ω-basis for A). Thus left alternativity is automatic.

However, the left Moufang axiom $L_{x(yx)} = L_x L_y L_x$ will only be satisfied for certain kinds of s. Indeed, $x^2 = q(x)1$ yields $x \circ y = q(x,y)1$ for $q(x,y) = q(x+y) - q(x) - q(y)$ $= 2\alpha\gamma + s(\beta)\delta + s(\delta)\beta = s(\beta)\delta + s(\delta)\beta$, so $L_{x(yx)} = L_{x(xy+yx)} - L_{x^2y}$ (left alternativity) $= L_{q(x,y)x} - L_{q(x)y}$ $= q(x,y)\alpha I + L_{q(x,y)\beta u} - q(x)\gamma I - L_{q(x)\delta u}$. On the other hand $L_x L_y L_x = L_x(L_y \circ L_x) - L_x^2 L_y = q(x,y)L_x - q(x)L_y$ $= q(x,y)\alpha I + q(x,y)L_{\beta u} - q(x)\gamma I - q(x)L_{\delta u}$. Thus the axiom reduces to $L_{q(x)\delta u} = q(x)L_{\delta u}$ (and its linearization $L_{q(x,y)\beta u}$ $= q(x,y)L_{\beta u}$). Since $L_{\omega u} \cong \begin{pmatrix} 0 & s(\omega) \\ \omega & 0 \end{pmatrix}$ the condition becomes $s(q(x)\delta) = q(x)\ s(\delta)$ (and its linearization $s(q(x,y)\delta) = q(x,y)s(\delta)$).

In particular s commutes with $q(\omega 1) = \omega^2$, $q(\omega u) = \omega s(\omega)$, $q(u) = s(1) = \sigma$, $q(u, \omega u) = \sigma\omega + s(\omega)$, so that $\{\sigma\omega + s(\omega)\}s(\omega)$ $= s(\{\sigma\omega + s(\omega)\}\omega) = s(\{\omega^2\sigma + \omega s(\omega)\}1) = \{\omega^2\sigma + \omega\sigma(\omega)\}s(1)$ $= \omega^2\sigma^2 + \sigma\omega s(\omega)$. Comparing gives $s(\omega)^2 = \omega^2\sigma^2$. In characteristic 2 this implies $\{s(\omega) - \sigma\omega\}^2 = 0$, so if Ω has no nilpotent elements $s(\omega) = \sigma\omega$ for all ω. ∎

5.3 Lemma. $A(\Omega,\sigma)$ is a left alternative division algebra iff (i) Ω is a field, (ii) $s - L_{\omega^2}$ is bijective on Ω for all ω.

Proof. Certainly Ω must be a field: it is a commutative associative subalgebra of A, and if $\alpha + \beta u$ is an inverse in A of $\omega \in \Omega$ then α is an inverse of ω in Ω .

Assume from now on Ω is a field. "Division algebra" means all L_x, R_x for $x \neq 0$ are bijective. Now L_x is bijective iff $L_x^2 = q(x)I$ is bijective, so the condition that all L_x for $x \neq 0$ be bijective is that $q(x) \neq 0$ for $x \neq 0$. In particular, this is satisfied when (i) and (ii) hold: clearly $q(x) = \alpha^2 + \beta s(\beta) = 0$ is impossible for $\beta = 0$ (since then $\alpha \neq 0$), while if $\beta \neq 0$ it would imply $s(\beta) = -\alpha^2/\beta = (\alpha/\beta)^2\beta$ (characteristic 2!) and therefore $s - L_{\omega^2}$ is not bijective for $\omega = \alpha/\beta$ since it kills β .

Turning now to the R's, if $x = \alpha + \beta u$ has $\beta = 0$ then $x = \alpha 1$ and $R_x = \alpha I$ is clearly bijective. If $\beta \neq 0$ then $x = \beta(\beta^{-1}\alpha 1 + u) = \beta y$, so $R_x = \beta R_y$ is bijective iff R_y is for $y = \omega 1 + u$. But

$$R_y = \omega I + R_u \simeq \begin{pmatrix} \omega & s \\ 1 & \omega \end{pmatrix} \text{ has } R_y^2 \simeq \begin{pmatrix} \omega^2+s & \omega s+s\omega \\ 2\omega & s+\omega^2 \end{pmatrix} = \begin{pmatrix} s-\omega^2 & \omega s+s\omega \\ 0 & s-\omega^2 \end{pmatrix}$$

in characteristic 2, which is invertible iff $s - L_{\omega^2}$ is invertible. ∎

5.4 Theorem. If Ω is a field of characteristic 2, with nontrivial involution *, and $\alpha = \alpha^*$ is a symmetric nonsquare in Ω, then $A(\Omega,s)$ for $s = * + L_\alpha$ is a left alternative division algebra which is not left Moufang.

Proof. Such an algebra is not left Moufang by 5.2 because $s \neq L_\sigma$: $\sigma = s(1) = 1^* + \alpha = 1 + \alpha, L_\sigma = I + L_\alpha \neq * + L_\alpha = s$ since $* \neq I$ is nontrivial by hypothesis.

Since Ω is a field, $A(\Omega,s)$ will be a division algebra by 5.3 as soon as all $s - L_{\omega^2} = * + L_{\alpha-\omega^2}$ are bijective, ie. $* + L_\delta$ is bijective for all $\delta = \alpha - \omega^2$. Now $(* + L_\delta)(* - L_{\delta *})$ $= (* - L_{\delta *})(* + L_\delta) = I - L_{\delta\delta *} = L_{1-\delta\delta *}$ (recall $* L_\omega = L_{\omega *} *$) is invertible since $1 - \delta\delta^*$ is nonzero in Ω no matter what $\delta = \alpha - \omega^2$ we chose: if ω is symmetric, $\omega^* = \omega$, then $\delta^* = \delta$ since $\alpha^* = \alpha$ by hypothesis, so $\delta\delta^* = 1 \Rightarrow \delta^2 = 1 \Rightarrow \delta = 1$ $\Rightarrow 1 = \alpha - \omega^2 \Rightarrow \alpha = 1 + \omega^2 = (1 + \omega)^2 \in \Omega^2$ (heavily using characteristic 2), contrary to our choice of α as a nonsquare, while if ω is nonsymmetric, $\omega + \omega^* \neq 0$, then $1 = \delta\delta^*$ $= (\alpha - \omega^2)(\alpha - \omega^{*2}) = \alpha^2 + (\omega\omega^*)^2 - \alpha(\omega + \omega^*)^2$ would imply $\alpha(\omega + \omega^*)^2 \in \Omega^2$, hence $\alpha \in \Omega^2$ (recall $\omega + \omega^* \neq 0$), again contrary to choice. Thus $1 - \delta\delta^* \neq 0$ is invertible in all cases, so all $s - L_{\omega^2} = * + L_\delta$ are bijective. ∎

5.5 <u>Example.</u> As a specific example, take $\Omega = \Phi(x)$ for any field Φ of characteristic 2. Then $f(x)^* = f(\frac{1}{x})$ is a nontrivial involution on Ω, $\alpha = x + 1/x$ is symmetric but a nonsquare (since $\alpha = x(1 + 1/x)^2$ and x is a nonsquare by $\Omega^2 = \Phi^2(x^2)$.

This particular Ω and $\sigma = * + L_\alpha$ lead to a left alternative division ring $A(\Omega,\sigma)$ which is not left Moufang. ∎

Exercises

5.1 Show $\Omega \xrightarrow{s} \Omega$ satisfies $s(q(x)\omega) = q(x)s(\omega)$ for all $q(x) = \alpha^2 + \beta s(\beta)$ iff (i) $s(\alpha^2 \omega) = \alpha^2 s(\omega)$, (ii) $s(s(\omega)) = \sigma^2 \omega$, (iii) $\{s(\omega) - \sigma\omega\}^2 = 0$.

5.2 Show directly R_x is invertible for $x = \beta u$ iff s is bijective; if $\alpha \neq 0$ show $x = \alpha y$, where $R_y = I + R_{\gamma u}$ is bijective iff $s + L_\gamma^{-2}$ is bijective.

§6 Homotopes

Given an element u in a left Moufang algebra A we can endow A with a new multiplicative structure $A^{(u)}$, the u-homotope of A, by

$$(6.1) \qquad x\cdot_u y = x(uy) .$$

Since $L_x^{(u)} = L_x L_u$ and $U_x^{(u)} y = x\cdot_u(y\cdot_u x) = (L_x L_u)(L_y L_u)x = L_x L_{U(u)y} x = U_x(U_u y)$ we see $A^{(u)}$ is again left Moufang:

$$L^{(u)}_{x^{2(u)}} = L_{x(ux)} L_u = L_x L_u L_x L_u = L_x^{(u)} L_x^{(u)}$$

$$L^{(u)}_{U^{(u)}(x)y} = L_{U(x)U(u)y} L_u = L_x L_u L_y L_u L_x L_u = L_x^{(u)} L_y^{(u)} L_x^{(u)} .$$

More elegantly, the homomorphism $A^+ \xrightarrow{L} (\mathrm{End}\ A)^+$ induces a homomorphism $A^{+(u)} \xrightarrow{L} (\mathrm{End}\ A)^{+(Lu)}$, where $A^{+(u)} = A^{(u)+}$, and $T \xrightarrow{F} TL_u$ is a homomorphism $(\mathrm{End}\ A)^{(Lu)} \longrightarrow \mathrm{End}\ A$ as $F(T\cdot_{L_u} S) = (TL_u S)L_u = F(T)F(S)$, so $x \rightarrow L_x L_u$ is a homomorphism $A^{(u)+} \longrightarrow (\mathrm{End}\ A)^+ = (\mathrm{End}\ A^{(u)})^+$.

If u is invertible we call $A^{(u)}$ the u-isotope of A; here

$$(6.2) \qquad 1^{(u)} = u^{-1}$$

is the unit for

$A^{(u)}$ since $L^{(u)}_{u^{-1}} = L_{u^{-1}} L_u = I$ and $R^{(u)}_{u^{-1}} = R_{uu^{-1}} = I$.

We have transitivity of homotopes

$$(6.3) \qquad \{A^{(u)}\}^{(v)} = A^{(U_u v)}$$

since $L_x^{(u)(v)} = L_x^{(u)} L_v^{(u)} = L_x L_u L_v L_u = L_x L_{U(u)v} = L_x^{(U_u v)}$,

and therefore symmetry in the case of isotopes:

$$\{A^{(u)}\}^{(u^{-2})} = A . \tag{6.4}$$

From this we can conclude that A is alternative iff its isotope $A^{(u)}$ is alternative. For if A is alternative so is any left homotope $A^{(u)}$, and if $A^{(u)}$ is alternative so is its left homotope A.

§7 Degree 2 Theorem

According to our general definition, A is of <u>degree two</u> over Φ if

$$(7.1)\quad x^2 - t(x)x + n(x)1 = 0 \qquad (t(1) = 2,\ n(1) = 1)$$

for linear t and quadratic n. A left alternative degree 2 algebra is automatically left Moufang, indeed even alternative.

(7.2) (Degree 2 Theorem) A left alternative algebra of degree 2 over a field is alternative.

Proof. To prove A is alternative we need to establish flexibility. Now $[L_x,R_x]y = [L_x,V_x]y = x(x\circ y) - x\circ xy$
$= x\{t(x)y + t(y)x - n(x,y)1\} - \{t(x)xy + t(xy)x - n(x,xy)1\}$
$= t(y)\{t(x)x - n(x)1\} - n(x,y)x - t(xy)x + n(x,xy)1$ by linearized (7.1),

$$[L_x,R_x]y = \{n(x,xy) - t(x^*(xy))\}1 - \{n(x,y) - t(x^*y)\}x\ .$$

Thus flexibility will follow if we can show the bilinear form

$$(7.3)\qquad f(x,y) = n(x,y) + t(xy) - t(x)t(y) = n(x,y) - t(x^*y)$$

vanishes identically. Since f vanishes when x or y is 1 ($n(x,1) = t(x)$ follows by linearizing $x \to x,1$ in (7.1) and using $t(1) = 2$), and also when $x = y$ (taking traces of (7.1)) so we may assume $1,x,y$ are linearly independent.

First suppose xy is linearly dependent on $1,x,y$. Then so is $yx = x\circ y - xy$,

$$(7.4)\qquad \begin{array}{l} xy = \alpha 1 + \beta x + \gamma y \\ \\ yx = \alpha' 1 + \beta' x + \gamma' y \end{array} \qquad \begin{cases} \alpha + \alpha' = -n(x,y) \\ \beta + \beta' = t(y) \\ \gamma + \gamma' = t(x) \end{cases} .$$

From $t(x)xy - n(x)y = x^2y = x(xy) = \alpha x + \beta x^2 + \gamma xy$ we can (by independence of 1,x,y) identify coefficients of x to see $t(x)\beta = \alpha + \beta t(x) + \gamma\beta$. Thus $\alpha + \beta\gamma = 0$, and dually with x and y interchanged

$$(7.5)\qquad \alpha + \beta\gamma = \alpha' + \beta'\gamma' = 0 .$$

Taking traces of (7.4) yields $t(xy) = 2\alpha + \beta t(x) + \gamma t(y)$ $= 2\alpha + \beta(\gamma + \gamma') + \gamma(\beta + \beta') = 2\alpha + \beta\gamma + (\beta + \beta')(\gamma + \gamma') - \beta'\gamma'$ $= t(y)t(x) + \{\alpha + \beta\gamma\} + \alpha - \{\alpha' + \beta'\gamma'\} + \alpha' = t(x)t(y) + \alpha + \alpha'$ (by (7.5)) $= t(x)t(y) - n(x,y)$ as required in (7.3).

Now suppose xy is independent of 1,x,y. We have the usual U-formula

$$(7.6)\qquad U_a\, b = n(a,b^*)a - n(a)b^* \in \Phi 1 + \Phi a + \Phi b$$

since $a(ba) = a(a\circ b) - a(ab) = a\{t(a)b + t(b)a - n(a,b)1\} - a^2b$ $= t(b)a^2 - n(a,b)a + n(a)b = \{t(a)t(b) - n(a,b)\}a + n(a)\{b - t(a)1\}$ by (7.1) and left alternativity. Then $0 = y\{x^2y\} - y\{x(xy)\}$ $= U_y\, x^2 - U_{y,xy}\, x + (xy)\{xy\} \in \{\Phi 1 + \Phi y + \Phi x^2\} - n(xy,x^*)y$ $- n(y,x^*)xy + t(xy)xy - n(xy)1 \subset \{t(xy) - n(x^*,y)\}xy + \Phi 1 + \Phi x + \Phi y$, so by independence the coefficient $f(x,y)$ of xy must be zero.

Thus $f(x,y)$ vanishes whether xy is dependent or independent of 1,x,y, and by (7.3) A is flexible. ∎

Exercises

7.1 Show that a left alternative degree 2 algebra over an arbitrary ring of scalars Φ is left Moufang.

7.2 Show that if A is left alternative of degree 2 so is any isotope $A^{(u)}$, with $t^{(u)}(x) = n(u^*,x)$ and $n^{(u)}(x) = n(u)n(x)$.

7.3 If $n(x,y)$ vanishes identically on A of degree 2, show A is commutative of characteristic 2; otherwise show (over a field) some isotope $A^{(u)}$ has nonzero trace $t^{(u)} \neq 0$.

7.4 If A is degree 2 over an algebraically closed field Φ with nondegenerate norm form $n(x,y)$, show either $A = \Phi 1$ or A contains a proper idempotent $e \neq 0,1$.

7.5 If A of degree 2 over an algebraically closed field Φ contains a proper idempotent $e_o \neq 0$, show for each x there are infinitely many $\lambda \in \Phi$ with $y = x + \lambda e_o$ separable, so if $[y,A,y] = 0$ for all separable y then $[x,A,x] = 0$ for all x. If $y = \alpha e + \beta(1 - e)$ is separable, show $[y,A,y] = 0$ if $[e,A,e] = 0$ for the idempotent e. Conclude that if $[e,A,e] = 0$ for all idempotents e then A is alternative.

7.6 Show $[e,x,e] = 0$ for any x and any idempotent e in a degree 2 left alternative algebra.