Appendix II

General Structure Theory

§1 Alternative division algebras

This Appendix is devoted to the study of alternative algebras without any finiteness restrictions. We begin in this section by completely determining all alternative divi- algebras; they turn out to be either associative or Cayley algebras, just as before. Of course, the associative division algebras are not completely classified, but from the standpoint of alternative algebras we consider our task finished if we have reduced our problem to one about associative algebras.

We recall the Nucleus = Center Theorem III.1.10: if A is an alternative division algebra, then either A is associative or its nucleus and center coincide, N(A) = C (A). It might happen that the nucleus and center reduce to zero; this is made unlikely by the following striking result, which provides a supply of nuclear elements.

1.1 (Fourth Power Theorem) In an alternative algebra the fourth power of any commutator lies in the nucleus,

$$[x,y]^4 \in N .$$

When $[x,y]$ is not a zero divisor, already the second power lies in the nucleus, $[x,y]^2 \in N$.

Proof. Set $z = [x,y]$; we will show

(1.2) $$z[z^2,a,b] = [z^2,a,b]z = 0$$

for all a,b. If z is not a zero divisor this implies all

$[z^2,a,b] = 0$ and $z^2 \in N$, while in general by Middle Bumping it implies $[z^4,a,b] = z^2[z^2,a,b] + [z^2,a,b]z^2 = 0$.

By symmetry we prove only $z[z^2,a,b] = 0$ in (1.2). Since $z[z^2,a,b] = (xy)[z^2,a,b] - (yx)[z^2,a,b] = \{[x,y,[z^2,a,b]] + x(y[z^2,a,b])\} - \{[y,x,[z^2,a,b]] + y(x[z^2,a,b])\} = 2[x,y,z\circ[z,a,b]] + x\{y[z^2,a,b]\} - y\{x[z^2,a,b]\}$ (by Middle Bumping), it suffices to prove

$$(1.3) \qquad [x,y,z\circ a] = z\circ[x,y,a] = 0$$

$$(1.4) \qquad x\{y[z^2,a,b]\} = y\{x[z^2,a,b]\}$$

for all a,b.

For (1.3), $[x,y,[x,y]\circ a] = [x,y,[x,y\circ a] - y\circ[x,a]]$ ($Ad\ x$ is a Jordan derivation by IV.3.24) $= -[x,y\circ a,[x,y]] - y\circ[x,y,[x,a]]$ (linearizing $y \to y, y\circ a$ in $[x,y,[x,y]] = 0$ by Artin, and noting $A_{x,y}$ is a Jordan derivation by IV.3.25 which kills y) $= -y\circ[x,a,[x,y]] + y\circ[x,a,[x,y]]$ (since $A_{x,[x,y]}$ is a Jordan derivation killing y, and linearizing $y \to y,a$ in $[x,y,[x,y]] = 0$) $= 0$. The second part of (1.3) follows since $A_{x,y}$ is a Jordan derivation killing $z = [x,y]$.

From this

$$(1.5) \qquad [z^2,x,b] = 0, \quad [z^2,y,b] = 0$$

since $[[x,y]^2,x,b] = -[[x,y]\circ[x,b],x,y]$ (linearizing $y \to y,b$ in $[[x,y]^2,x,y] = 0$) $= -[z\circ[x,b],x,y] = 0$, consequently

$$(1.6) \qquad x[z^2,a,b] = [z^2,a,bx], \quad y[z^2,a,b] = [z^2,ay,b]$$

because $x[z^2,a,b] = -x[[x,y]\circ[a,y],x,b]$ (linearize $x \to x,a$

in (1.5)) = $[x, [x,y]\circ[a,y], bx]$ (left bumping) = $-[a, [x,y]^2, bx]$ (linearize $x \to x,a$ in (1.5)) = $+[z^2,a,bx]$.

From (16.) we get (1.4) by the HIDING TRICK: $x\{y[z^2,a,b]\} = x[z^2,ay,b] = [z^2,ay,bx] = y[z^2,a,bx] = y\{x[z^2,a,b]\}$. This finishes (1.4) and the Theorem. ∎

Thus supplied with central elements we can establish the structure of an arbitrary alternative division algebra.

1.7 (Bruck-Kleinfeld-Skornyakov Theorem) An alternative division algebra is either associative or a Cayley algebra over its center.

Proof. Assume throughout that the alternative division algebra A is <u>not</u> associative. We wish to show A is a degree 2 algebra over its center. But for any element x we can actually write down a quadratic equation it satisfies:

(1.8) $$\alpha x^2 - \beta x + \gamma 1 = 0 \quad \text{(Hall's identity)}$$

where
$$\alpha = [x,y]^2$$
$$\beta = [x,y]^2 x - [x,y]x[x,y] = [x,y]\circ[x,y']$$
$$\gamma = [x,y]x[x,y]x = [x,y']^2$$

(here $y' = yx$ has $[x,y'] = xyx - yxx = [x,y]x$). By Artin's theorem the above equation holds identically in x and y. We must show the coefficients lie in the center and can be chosen nontrivial.

Since a division algebra has no zero divisors, $\alpha = [x,y]^2$

and $\gamma = [x,y']^2$ lie in the nucleus N by the Fourth Power Theorem, and so does the linearization $\beta = [x,y]\circ[x,y']$. By the Nucleus = Center Theorem and the assumed nonassociativity of A, α,β,γ lie in $N = C$.

Thus every element x satisfies a quadratic equation over C. The only trouble is that it might be a trivial equation. It will be nontrivial if $[x,y] \neq 0$ (since then $\alpha = [x,y]^2 \neq 0$), so by proper choice of y we can get a nontrivial equation unless $[x,y] = 0$ for all possible y. But such an x already lies in C by the Commutativity-Implies-Centrality Lemma III.4.1 (since A has no nilpotents), in which case $x = \delta$ satisfies $x^2 - 2\delta x + \delta^2 = 0$.

So far we know every x satisfies an equation

(1.9) $$x^2 - t(x)x + n(x)1 = 0$$

for some coefficients $t(x), n(x)$ in the center C. If we can show t is linear then A will be degree 2 over C; since A is already semiprime (being a division algebra), by the Equivalence and Hurwitz Theorems II.2.14 and II.4.1, A will have to be a Cayley algebra over C.

But in II.1.6 we saw that (1.9) and the fact that A has no zero divisors force it to be degree 2. ∎

1.10 Remark For a different proof of the last statement, recall that we have seen t is automatically linear if $|C| > 2$. But if $C = \mathbb{Z}_2$ is finite then any two elements $x,y \in A$ generate a finite associative subalgebra $C[x,y] = C1 + Cx + Cy + Cxy = C1 + Cx + Cy + Cyx$ (note $xy + yx = (x+y)^2 - x^2 - y^2 \in C(x+y) + Cx + Cy + C1$) without zero divisors, which must be a finite field by Wedderburn's Theorem. In particular, x and y commute. This holds for all x and y, so A is commutative, hence associative by III.4.1, contrary to our hypothesis. ∎

The preceding remark once again shows

1.11 (Wedderburn Theorem) Any finite alternative division ring is a finite (commutative, associative) field. ∎

#AII.1 Problem Set on Domains

Go back through the proof of the Bruck-Kleinfeld-Skornyakov Theorem, making whatever additional arguments are necessary, to establish

(Theorem) If A is an alternative algebra without zero divisors, then A is either associative or an order in a Cayley algebra.

Suggested steps are:

1. Show that if A is not associative it is of degree 2 over $C = N$, and semiprime.
2. Show C is an integral domain with quotient field $\tilde{C}$, and that A can be imbedded as a C-order in its central closure $\tilde{A} = A \otimes_C \tilde{C}$. Show $\tilde{A}$ has no zero divisors, and is of degree 2 over $\tilde{C}$ iff is degree 2 over C.
3. If $\tilde{A}$ is not associative but is a domain of degree 2 over a field $\tilde{C}$, show $\tilde{A}$ is a Cayley algebra, so our original A is an order in a Cayley algebra.

#AII.2 Problem Set on the Fourth Power Theorem

1. Prove that $[x,m][y,z,w][x,n] = [x,m][x,n][y,z,w] = 0$ for all $x,y,z,w \in A$ and $n,m \in N(A)$.
2. Prove any associator $a = [x,y,z]$ satisfies $a^2 = [x,y,az] - c(az) = -B_{x,y}(az)$ for $c = [x,y]$.
3. Prove that $[[x,y]^2,z,w]^2 = 0$ holds in all alternative algebras.
4. Conclude that if A has no nilpotent elements, the square $[x,y]^2$ of any commutator lies in the nucleus.

This improves on the Fourth Power Theorem (instead of no zero divisors we need only no nilpotents). However, we cannot generalize the Nucleus = Center Theorem from the case of no zero divisors to the case of no nilpotents: if D is a central associative division algebra and C a Cayley division algebra then $A = D \boxplus C$ has no nilpotents but has nucleus $N(A) = D \oplus \Phi$ different from A or $C(A) = \Phi \oplus \Phi$.

§2 Simple alternative algebras

In the section we extend the Bruck-Kleinfeld-Skornyakov Theorem to arbitrary simple algebras. The fact that we are no longer in a division algebra forces us to modify slightly the approach of the previous section.

The first order of business is to prove that in a simple not-associative algebra A the nucleus N and center C coincide. The proof of the Nucleus = Center Theorem required cancellation, so we will give a different (and more general) proof.

2.1 (Prime Nucleus Theorem). If A is a prime alternative algebra either A is associative, $N(A) = A$, or the nucleus and center coincide, $N(A) = C(A)$.

Proof. Assume $N(A) \neq C(A)$, so N doesn't commute with everything: $[A, N] \neq 0$. We will show A is associative.

$B = \hat{A}[A, N]$ is an ideal since $AB = (A\hat{A})[A, N]$ (recall $[A, N] \subset N$ by III.1.8) $\subset B$ and $BA = \hat{A}[[A, N], A] + \hat{A}A[A, N] \subset \hat{A}[N, A] + A[A, N] \subset B$. By hypothesis, B is nonzero; therefore by primeness if $aB = 0$ or $Ba = 0$ then $a = 0$ ($\mathrm{Ann}_L(B)B = 0$ and $B\,\mathrm{Ann}_R(B) = 0$ force the ideals $\mathrm{Ann}_L(B)$ and $\mathrm{Ann}_R(B)$ to vanish by primeness).

But if x commutes with N then any $[x,y,z]$ belongs to $\mathrm{Ann}_R(B)$, $\hat{A}[A, N][x,y,z] = 0$, since for any $s = [a,n]$ we have $s[x,y,z] = [sx,y,z]$ $(s \in N) = 0$ $(sx = anx - nax = axn - nax = [ax,n] \in [A, N] \subset N$ if x commutes with N).

In particular, this holds when x is an associator because the nucleus commutes with associators (see III.1.7). Therefore $[x,y,z] = 0$ for all associators x and arbitrary y,z, ie. x lies in the nucleus. Thus all associators lie in the nucleus:

$$[A, A, A] \subset N . \tag{2.2}$$

Then for arbitrary x,y,z the nuclear element $n = [x,y,z]$ has $n[x,z,b] = [nx,a,b] = 0$ since $nx = [x,y,z]x = [xy,z,x] \in N$ by (2.2) and right bumping,

$$[x, A, A][x, A, A] = 0 . \tag{2.3}$$

Consider an associator $n = [x,y,z]$ lying in N. If $[A, n] = 0$ then n lies in C, otherwise $S = [A, n] \neq 0$. But S kills n since $[a,n]n = [a,n][x,y,z] = -[x,n][a,y,z]$ (linearizing III.1.9) $= nx[a,y,z]$ ($xn[a,y,z] = 0$ by (2.3) since n and $[a,y,z]$ have common factor z) $= [xy,z,x][a,y,z]$ (left bumping on n) $= 0$ by (2.3) (again because of a common factor z in both associators). In this case primeness forces $n = 0$ since $S^{\perp} = \mathrm{Ann}_R(S)$ is an ideal (even though S isn't: $S^{\perp}A$ kills S because $S(S^{\perp}A) = (SS^{\perp})A = 0$ by nuclearity of S, $AS^{\perp}$ kills S because $S(AS^{\perp}) = [A,n]AS^{\perp} = \{[AA, n] - A[A, n]\}S^{\perp} \subset \hat{A}[A, n]S^{\perp} = 0$), and a nonzero ideal $S^{\perp}$ could not kill $S \neq 0$. In either case n lies in C. It also has square zero by (2.3), hence generates a nilpotent ideal $\hat{A}n = n\hat{A}$; primeness then forces $n = 0$. Therefore all associators $n = [x,y,z]$ vanish, and a is associative. ∎

We now are set to prove

(2.4) (Kleinfeld's Simple Theorem) A simple alternative algebra is either associative or a Cayley algebra over its center.

Proof. We may regard the simple algebra A as an algebra over its centroid Γ (which is a field containing the center C of A, and coinciding with it if $C \neq 0$). Since any scalar extension $A_\Omega = A \otimes_\Gamma \Omega$ remains simple, and since if A_Ω is associative or Cayley then A was to begin with, it suffices to prove the result for A_Ω. Taking an infinite Ω (in case Γ was finite), we may assume the centroid Γ of A is infinite. This will allow us to make use of the Zariski topology on A.

Assume throughout that A is not associative. To show such an A is Cayley, we begin as in the last section from Hall's identity

$$\alpha x^2 - \beta x + \gamma = 0 \tag{2.5}$$

($\alpha = [x,y]^2$, $\beta = [x,y] \circ [x,y']$, $\gamma = [x,y']^2$ for $y' = yx$). Since A is no longer a division algebra we do not know $z^2 \in N$ for all commutators $z = [x,y]$, so we have no guarantee α, β, γ lie in Γ.

However, we claim that if $z = [x,y]$ is invertible then we do have $\alpha = z^2$, $\beta = z \circ z'$, $\gamma = z'^2$ lying in Γ for arbitrary $z' = [x,y']$. Indeed, linearizing $y \to y + \lambda y'$ in (1.2) yields

$$z[z^2,a,b] = 0$$

$$z[z\circ z',a,b] + z'[z^2,a,b] = 0$$

$$z[z'^2,a,b] + z'[z\circ z',a,b] = 0$$

for any a,b. If z is invertible we can cancel it from the first relation to get $[z^2,a,b] = 0$ for all a,b; the second relation then becomes $z[z\circ z',a,b] = 0$, and we can again cancel z to get $[z\circ z',a,b] = 0$; the third relation then reduces to $z[z'^2,a,b] = 0$, whence $[z'^2,a,b] = 0$ by cancellation. Consequently $[z^2,a,b] = [z\circ z',a,b] = [z'^2,a,b] = 0$ for all a,b and $z^2, z\circ z', z'^2$ lie in N. Since A is prime and by hypothesis not associative, the Prime Nucleus Theorem assures us N = C, so that α,β,γ lie in Γ (and α is invertible since $\alpha = z^2$).

So far we know x will satisfy an equation of degree 2 over Γ as soon as some $z = [x,y]$ is invertible. Furthermore, since $z^4 \in N = C$ by the Fourth Power Theorem, and $C \subset \Gamma$ is a field, z will be invertible as soon as $z^4 \neq 0$.

Thus x will satisfy a nontrivial quadratic equation if we can just find a y such that $[x,y]^4 \neq 0$. Since A is not a division algebra there is no reason to expect very many such pairs x,y. The amazing thing is that if we can find <u>one</u> such pair x_o,y_o we will be done! Recall that we are trying to prove <u>every</u> x is quadratic. Being quadratic just means $x^2,x,1$ are linearly dependent over Γ, ie. $1\wedge x\wedge x^2 = 0$ in the exterior algebra

$\Lambda(A)$ over Γ. But $F(x) = 1 \wedge x \wedge x^2$ defines a polynomial map from A into $\Lambda(A)$. Further, by our remarks above $F(x) = 0$ whenever $[x,y_o]^4 \neq 0$. But then F vanishes on the set of x for which $G(x) = [x,y_o]^4 \neq 0$; this set is Zariski-open since it is defined by a polynomial equation, and it is non-empty since $G(x_o) \neq 0$ by choice of x_o, y_o, so because we made sure we were working over an infinite field Γ we can conclude the set is Zariski-dense. If the polynomial map F vanishes on a Zariski-dense set it vanishes everywhere, and all x are quadratic.

At this stage we have established that if we can find a single pair x_o, y_o for which $[x_o, y_o]^4 \neq 0$ we will know all elements are quadratic. How could we possibly fail? Only if $[x_o, y_o]^4 = 0$ vanished identically. Now the polynomial identity $p(x,y) = [x,y]^4 = 0$ is not satisfied by matrix algebras $M_n(\Omega)$ of degree $n \geqslant 2$: if we set $x = e_{12}$, $y = e_{21}$ then $[x,y] = e_{11} - e_{22}$ has $[x,y]^4 = e_{11} + e_{22} \neq 0$. By amazing coincidence we just happen to have proved a theorem about this situation in the previous appendix: from Lemma 0 of Appendix I we know that if $[x,y]^4 = 0$ identically then either A contains locally nilpotent ideals (which is impossible for a simple A; see VI.5.10), or else is commutative and associative (which contradicts our hypothesis). Consequently our A refuses to obey the law $[x,y]^4 = 0$, and somewhere we must be able to find $[x_o, y_o]^4 \neq 0$.

By now we know each x satisfies a nontrivial equation over Γ of degree 2; since Γ is infinite we conclude A is degree 2 over Γ (see II. 1.6). A is certainly semiprime, so as in the last section the Equivalence and Composition Algebra Theorems show it must be Cayley. ∎

2.6 Remark. An alternate proof would not seek to prove that all elements x are quadratic, only that there exists at least one x which is separable quadratic. For if such an element exists we can apply Albert's More General Theorem to deduce A is associative or Cayley. As before x will be quadratic, $\alpha x^2 + \beta x + \gamma = 0$ for $\alpha, \beta, \gamma \in \Gamma$, as soon as $[x,y]^4 \neq 0$. We want a separable equation, $\beta^2 - 4\alpha\gamma \neq 0$, ie. the discriminant $\delta = \{[x,y] \circ [x,yx]\}^2 - 4[x,y]^2[x,yx]^2 \neq 0$. The only way we could fail to find a separable quadratic element is that for each x,y either $[x,y]^4 = 0$ or $\delta = 0$; in that case the product

$$q(x,y) = [x,y]^4 \{([x,y] \circ [x,yx])^2 - 4[x,y]^2[x,yx]^2\}$$

would vanish identically on A. But once again $q(x,y)$ is not satisfied by any $M_n(\Omega)$ for $n \geq 2$: if $x = e_{11}$, $y = e_{12} - e_{21}$ then $yx = -e_{21}$, $[x,y] = e_{12} + e_{21}$, $[x,yx] = e_{21}$ so that $[x,y]^2 = [x,y]^4 = [x,y] \circ [x,yx] = e_{11} + e_{22}$, $[x,yx]^2 = 0$ and $q(x,y) = e_{11} + e_{22} \neq 0$. As before, the Lemma shows $q(x,y) \equiv 0$ is impossible if A is neither locally nilpotent nor associative, so we must be able to find $q(x_o,y_o) \neq 0$ and consequently a separable x_o. ∎

§AII.3 Problem Set on Reasonable Elements

Those who don't believe in the Zariski topology may give a "constructive" proof. Say x,y is a <u>reasonable pair</u> if $[x,y]^4 \neq 0$, and say x is <u>reasonable</u> if there is a y such that x,y is a reasonable pair. Assume that A is an alternative algebra over a field Φ with more than 5 elements, $|\Phi| > 5$.

1. Show that if x,y_o is reasonable and y arbitrary, there are at least two values of $\lambda \in \Phi$ for which $x,y + \lambda y_o$ is reasonable.

   From now on assume x,y reasonable $\Rightarrow [x,y]^2 \in \Phi 1$.

2. Show that if x is reasonable then all $[x,z]^2$ and $[x,z]\circ[x,w]$ lie in $\Phi 1$.
3. Show all reasonable x are quadratic.
4. Show that if x_o,y_o is reasonable then any x satisfies relations $x^2 = \alpha 1 + \beta x + \gamma x_o$ and $x^2 = \delta 1 + \varepsilon x + \nu y_o$.
5. Show in all cases x is quadratic, $x^2 \in \Phi 1 + \Phi x$.
6. Prove the <u>Theorem</u>. If A is an alternative algebra over a field Φ with more than 5 elements such that

   (i) there exists at least one reasonable pair x_o,y_o

   (ii) x,y reasonable implies $[x,y]^2 \in \Phi 1$

   then A is degree 2 over Φ.
7. Deduce that all simple non-associative algebras which contain reasonable pairs are degree 2.

§3 Prime algebras

At only two places in Kleinfeld's Simple Theorem did we require simplicity of A as opposed to primeness: when we reduced to the case of an algebra over an infinite field, and when we concluded that a simple algebra couldn't contain locally nilpotent ideals. The first difficulty is easily taken care of, the second is not.

Note that the elements of the centroid $\Gamma(A)$ are injective on a prime algebra A, $\gamma a = 0$ implies $\gamma = 0$ or $a = 0$. Indeed, if γ is in the centroid then $\mathrm{Im}\,\gamma$ and $\mathrm{Ker}\,\gamma$ are ideals in A which kill each other, $\mathrm{Im}\,\gamma \cdot \mathrm{Ker}\,\gamma = 0$, since $(\gamma A)\cdot(\gamma^{-1}0) = A\cdot\gamma(\gamma^{-1}0) = A\cdot 0 = 0$. Thus if A is prime either $\mathrm{Im}\,\gamma = 0$ (and $\gamma = 0$) or $\mathrm{Ker}\,\gamma = 0$ (and γ is injective). In particular, Γ is an integral domain, so has a field of fractions $\tilde{\Gamma}$. Then $\tilde{A} = A \otimes_{\Gamma} \tilde{\Gamma}$ is an alternative $\tilde{\Gamma}$ algebra (called the <u>centroid closure</u> of A; its centroid contains the field of fractions $\tilde{\Gamma}$ of the original centroid Γ, and coincides with it if A is unital). Furthermore, $\tilde{A}$ is still prime and A is imbedded as an <u>order</u> in $\tilde{A}$ (a Γ-subalgebra such that $\tilde{\Gamma} A = \tilde{A}$). The easiest way to see these last two properties is to observe that $\tilde{A}$ is isomorphic to the algebra of "fractions" $^{a}/_{\gamma}$ (a in A, $\gamma \neq 0$ in Γ) with the usual algebraic operations obtained by the "standard construction", under the isomorphism $a \otimes {}^{\delta}/_{\gamma} \to {}^{\delta(a)}/_{\gamma}$ (note that if $\Sigma\, a_i \otimes {}^{\delta_i}/_{\gamma} = \Sigma\, \delta_i(a_i) \otimes {}^{1}/_{\gamma} \longrightarrow \{\Sigma\, \delta_i(a_i)\}/_{\gamma} = 0$ then $\Sigma\, \delta_i(a_i) = 0$, so the map is injective;

clearly it is a surjective algebra homomorphism). The fact that $\gamma a \neq 0$ if $\gamma \neq 0$ and $a \neq 0$ implies A is imbedded in this algebra of fractions, and primeness of the fraction algebra comes about because if $\tilde{B}$, $\tilde{C}$ are nonzero ideals of fractions with $\tilde{B}\tilde{C} = 0$ then $B = \tilde{B} \cap A$, $C = \tilde{C} \cap A$ are nonzero ideals in A satisfying $BC = 0$ (nonzero because $\tilde{\Gamma}B = \tilde{B}$: if $b/\gamma \in \tilde{B}$ then $b = \gamma \cdot b/\gamma \in \tilde{B} \cap A = B$ and $b/\gamma = 1/\gamma \cdot b \in \tilde{\Gamma}B$).

(We could do exactly the same thing with the center in place of the centroid, but the center may well be zero).

3.1 (Order Proposition) Every prime alternative algebra A with centroid Γ is imbedded as a Γ-order in its centroid closure $\tilde{A}$, which is a prime alternative algebra over the field of fractions $\tilde{\Gamma}$ of Γ. ∎

The question of the structure of prime algebras thus reduces to the structure of prime algebras over fields (modulo a description of all orders). This is all very well, but if the original centroid Γ was finite the same will be true of $\tilde{\Gamma}$ (indeed $\tilde{\Gamma} = \Gamma$!) We need a method of creating an infinite centroid without at the same destroying primeness: unlike simplicity, primeness is not preserved by arbitrary scalar extensions.

3.2 (Proposition) If A is a prime alternative algebra with centroid Γ then the algebra A[t] of polynomials in t is a prime alternative algebra with infinite centroid containing $\Gamma[t]$.

Proof. $A[t] = A\otimes_\Phi \Phi[t]$ remains alternative, and contains $\Gamma[t] = \Gamma\otimes_\Phi \Phi[t]$ in its centroid. It is prime since if $\tilde{B}\tilde{C} = 0$ for nonzero t-invariant ideals $\tilde{B},\tilde{C}$ in A[t] then BC = 0 where B,C are the (nonzero) ideals in A of leading coefficients of $\tilde{B},\tilde{C}$ (if b_n, b'_m lead $\tilde{b},\tilde{b}'$ in $\tilde{B}$ then $ab_n, b_n a, \alpha b_n + \beta b'_m$ are zero or the leading coefficients of $a\tilde{b}, \tilde{b}a, \alpha t^m\tilde{b} + \beta t^n\tilde{b}' \in \tilde{B}$, using t-invariance of $\tilde{B}$). ∎

Putting all these reductions together, we can prove

3.3 (Slater's Prime Theorem) If A is a prime alternative algebra then either

(i) A is a prime associative algebra

(ii) A is an order in a Cayley algebra over a field

(iii) A contains locally nilpotent ideals.

Proof. Start with A, form A[t] (with infinite centroid Γ_0) then $\tilde{A}$ (algebra over an infinite field $\tilde{\Gamma}$). The argument of Kleinfeld's Simple Theorem shows $\tilde{A}$ is either associative (whence its subalgebra A is associative), a Cayley algebra over its center (in which case A is an order in $\tilde{A}$: $\tilde{A} = \tilde{\Gamma}A[t] = \tilde{\Gamma}\Phi[t]A = \tilde{\Gamma}A$), or $\tilde{A}$ contains locally nilpotent ideals (in which case A does: if $\tilde{N}$ is locally nilpotent in $\tilde{A}$ then $N_o = \tilde{N} \cap A[t]$ is locally nilpotent in A[t] and N = {leading coefficients of N_o} in A, where $N \neq 0$ since $\tilde{N} = \tilde{\Gamma} N_o \subset \tilde{\Gamma}\Phi[t]N$). ∎

Note we cannot sharpen (iii) to say A itself is even nil: if B is a nil not-associative prime algebra (it is still open whether such exist - they would have to be of characteristic 3) then $\Phi 1 + B = A$ is still prime, not associative, not Cayley, but also no longer nil - it only contains the nil ideal B.

Of course, we can say a little more - namely that if A is not associative nor Cayley, the nil radical N consists of all nilpotent elements and A/N is commutative associative. Thus (as in our example) A is an extension of a nil algebra by a commutative associative algebra without nilpotent elements.

It is highly unlikely that there can be any of these prime nil algebras which are not associative (and, if the less likely Köthe Conjecture holds, there aren't even any simple nil associative algebras). Indeed, it can be shown that if such beasts exist there would exist an algebra A satisfying all of the following:

(i) A is nonzero but $3A = 0$ and $N(A) = 0$.

(ii) A satisfies the polynomial identity $[x,y]^4 = 0$.

(iii) A is prime yet locally nilpotent.

(iv) A is semiprime yet not strongly semiprime; there exist trivial elements z, $zAz = 0$.

(v) A contains no minimal one-sided ideals.

(vi) Any one sided ideal B of A is just as bad, possessing properties (i)-(vi).

It would be a sad day indeed if such a beast were ever uncovered. [See the bestiary in reference 00].

There are certain general conditions under which an algebra necessarily has a nucleus. Using the Fourth Power Theorem as a source of nuclear elements, we have

3.4 (Nuclear Existence Proposition) If $A \neq 0$ is a strongly semiprime alternative algebra then its nucleus is nonzero, $N(A) \neq 0$.

Proof. Assume to the contrary $N = 0$. Let $z = [x,y]$ be an arbitrary commutator. By the Fourth Power Theorem we know

$z^4 \in N$ and $[z^2, A, Az^2] = z^2[A, A, z^2]$ (left bumping) $= z\{z[A, A, z^2]\} = 0$ (recall (1.2)), hence by the Associator Derivation Formula IV. 3.12 all $[z^2, Az^2] \in N$. If $N = 0$ we get $z^4 = [z^2, az^2] = 0$, therefore $z^2az^2 = 0$ for all a. By our hypothesis that A has no trivial elements, $z^2 = 0$. Thus $[x,y]^2 = 0$ for all x,y, and by linearization $[x,y]\circ[x',y] = 0$ for all x,x',y.

Now let z be any element with $z^2 = 0$ and let a,b be arbitrary in A. We have $U_{zaz}\, b = U_{z[a,z]}\, b$ (if $z^2 = 0$) $= R_{[a,z]}\, U_z\, L_{[a,z]}\, b$ (Right Fundamental) $= \{z[a,z]\cdot bz\}[a,z]$ (Middle Moufang) $= \{z[a,z]\cdot[b,z]\}[a,z]$ (as $z[a,z]\cdot zb = zaz\cdot zb = z\{a(z^2b)\} = 0$ by Left Moufang) $= z\{[a,z][b,z][a,z]\}$ (Right Moufang) $= -z\{[b,z][a,z][a,z]\} = 0$ by the vanishing of squares of commutators $[a,z]^2 = [a,z]\circ[a',z] = 0$. Thus zaz is trivial, so $zaz = 0$ by hypothesis whenever $z^2 = 0$; but then z is trivial, so $z = 0$ whenever $z^2 = 0$.

This implies A contains no nilpotent elements. In particular, $[x,y]^4 = 0$ implies all commutators $[x,y] = 0$. Thus A is commutative without nilpotent elements, hence associative (by III. 4.1). Then $A = N(A) = 0$ contradicts $A \neq 0$, and the assumption $N = 0$ is untenable. ∎

The Kleinfeld Strong Semiprimeness Theorem says a semiprime algebra on which 3 is injective a surjective is necessarily strongly semiprime, so

3.5 (Theorem). A nonzero semiprime algebra on which 3 is injective or surjective has a nonzero nucleus. ■

This explains why the structure of prime algebras is known in characteristic $\neq 3$.

3.6 (Theorem). If A is a prime nilalgebra of characteristic $\neq 3$ then A is associative.

Proof. We may assume $A \neq 0$, and prime implies semiprime, so by the previous therem $N(A) \neq 0$. Suppose A is not associative; by the Prime Nucleus Theorem $C(A) = N(A) \neq 0$, so A is not nil. (If C(A) contained a nilpotent element it would contain $c \neq 0$ with $c^2 = 0$, and $\hat{A}c$ would be a trivial ideal, contrary to semiprimeness). ■

Exercise

3.1 Show that the center C of a prime ring is 0 or an integral domain, and in the latter case that the central closure $\tilde{A} = A \otimes_C \tilde{C}$ is isomorphic to the algebra of "fractions" a/c and has center just the field of fractions $\tilde{C}$ of C.

3.2 What is the relation of the centroid of the centroid closure $\tilde{A} = A \otimes_\Gamma \tilde{\Gamma}$ to the original centoid Γ ?

3.3 Give an example where the centroid closure $\tilde{A}$ is degree 2 over $\tilde{\Gamma}$ but A is not degree 2 over Γ .

3.4 Give an example of a prime associative algebra with center a field Φ and an extension $\Omega \supset \Phi$ such that A_Ω no longer remains prime.

3.5 Does an ideal $B \triangleleft A$ inherit primeness from A ? (It does in the associative case).

3.6 If $\mathbb{Z} \subset \Phi \subset \Gamma(A)$ show A is $\mathbb{Z}$-prime (prime as $\mathbb{Z}$-algebra, ie. as a ring) iff A is Φ-prime iff A is Γ-prime.

3.7 Show that if A is an order in a Cayley algebra $\tilde{A}$ over a field Ω, so is any one-sided ideal B of A.

#AII.4 Problem Set on Algebras with Non-nil Heart

As in associative algebras, the <u>heart</u> M of an alternative algebra A is the intersection of all nonzero ideals of A. (In particular, a non-trivial algebra A is simple iff it is all heart, $M = A$). If $M \neq 0$, it is the unique minimal ideal.

1. Show that if the heart M of A is not trivial, $M^2 \neq 0$, then A is prime.
2. Deduce that if the heart of A is not trivial, either A is associative or $N(A) = C(A)$.
3. If B is a nonzero ideal in a prime algebra, show $[B,A,C(B)] = 0$, $[A,A,C(B)] = 0$, $[C(B),A] = 0$. Conclude that if A is prime and $B \triangleleft A$ then $C(B) \subset C(A)$.
4. Show that in general if $c \in C(A)$ then $cM = 0$ or $cM = M$ (M the heart); if A is prime show $c\,M = M$ for all $c \in C(M)$ and hence $C(M)$ is zero or a field. If A is prime and $C(M)$ is a field with unit e, show e is the unit for A, hence $M = A$. Conclude that if A is prime and its heart M has nonzero center $C(M) \neq 0$, then $C(M)$ is a field and $A = M$ is simple.
5. Assume A is prime, not associative, and $C(M) = 0$. Show fourth powers of commutators in M are zero, so the nilpotent elements of M form an ideal $Z(M) \triangleleft M$. Show $Z(M) \triangleleft A$, conclude $Z(M) = M$ or $Z(M) = 0$. If $Z(M) = M$ then M is nil; if $Z(M) = 0$ show $M = C(M) = 0$. Conclude that if A is prime but not associative, with heart M satisfying $C(M) = 0$, then M is nil.
6. Prove the <u>Theorem</u>. If A has a non-nil heart then A is either associative or a Cayley algebra.

#AII.5 Problem Set on Primitive Algebras

An associative algebra is primitive if it has a faithful irreducible representation ϕ on M, in which case $M \cong A/B$ under $\psi: a \to am$ for $B = \text{Ker}\,\phi$ a maximal modular left ideal which contains no nonzero two-sided ideal.

Although the module definition does not go over, the ideal definition can be carried over to alternative algebras: an alternative algebra A is (left) primitive if it contains a left primitivity ideal (a maximal modular left ideal B with kernel $K(B) = L(A,B) = 0$; see V.2).

1. If $L(A,B) = 0$ show $B = 0$ if $BA \subset B$ and $A = B = 0$ if $B\hat{A} = A, B^2 = 0$ $(B \triangleleft_\ell A)$.
2. Show that for any prime (resp. semiprime) nonassociative algebra A with $A^2 \neq 0$, the center $C(A)$ and centroid $\Gamma(A)$ contain no zero divisors (resp. nilpotent elements).
3. Show $[x,y,A] = 0$ in an alternative algebra implies $[x,y] \in N(A)$ and $[x,y]x \in N(A)$. If $[x,y]$, $[x,y]x \in C(A)$ show $[x,y]^2 = 0$. Conclude that if A is an alternative algebra with $N(A) = C(A)$ then $[x,y,a] = 0$ implies $[x,y]$ is a nilpotent element of the center; if A is semiprime then $[x,y] = 0$.
4. Assume $N(A) = C(A)$ and $L(A,B) = 0$ for $B \triangleleft_\ell A$, A semiprime. Show B is commutative ($[B,B] = 0$), then $[B^2,A,A] = 0$, then B^2 is central ($[B^2,A] = 0$), then $B = 0$. Conclude that if A is semiprime with $N(A) = C(A)$ and B is a left ideal with $L(A,B) = 0$ then B is trivial, $B^2 = 0$.

We can put these results together to show that a general simple algebra (even nil of characteristic 3) must look something like a Cayley algebra.

5. Establish the <u>Theorem</u>. If A is simple but not associative it contains no proper one-sided ideals.

Now we return to primitivity.

6. If B is a modular left ideal and C a supplementary left ideal ($B + C = A$) then $aC \subset B \Rightarrow a \in B$. If B is maximal modular, C a (two-sided) ideal not contained in B, show $D\,C = 0$ for an ideal D implies $D \subset B$. Conclude that a primitive algebra is prime.

7. Show that if B is a left primitivity ideal for A then either $B = 0$ or $B\hat{A} = A$; conclude that $B = 0$ if $B^2 = 0$, in which case A has no proper left ideals and has a right unit; deduce A has a unit. Conclude that if B is a left primitivity ideal for A with $B^2 = 0$ then A is simple with unit.

Since we know all simple algebras with unit (they cannot be nil even in Characteristic 3), show

8. <u>Theorem</u> A primitive algebra is either associative or a Cayley algebra.

Note that this strengthens Slater's Prime Theorem when A is primitive rather than merely prime.

#AII.6 Problem Set on the Jacobson-Kleinfeld Radical

In the associative case the Jacobson radical can be defined either in terms of quasi-invertibility or in terms of primitivity. In the associative case these two approaches lead respectively to the Jacobson-Smiley radical and the Jacobson-Kleinfeld radical; it is not clear they coincide.

Define an ideal K to be primitive in A if A/K is primitive as an algebra.

1. Show K is primitive iff $K = L(\hat{A}, B)$ for some maximal modular left ideal B.
2. Conclude the intersection $\bigcap B$ of all maximal modular left ideals B contains the intersection $\bigcap K$ of all primitive ideals K .
3. If $K = L(\hat{A}, B)$ is primitive show $\bar{B}$ is a maximal modular left ideal in $\bar{A} = A/K$. If $\bar{A}$ is Cayley show $B = K$ and hence $K \supset \bigcap B$. If $\bar{A}$ is primitive associative show $\bigcap \bar{B} = \bar{0}$; conclude $K \supset \bigcap B$. Conclude that $\bigcap K$ contains $\bigcap B$.
4. Theroem The intersection of all maximal modular left ideals coincides with the intersection of all (left) primitive ideals of A.

This intersection is called the Jacobson-Kleinfeld radical JK(A) of A. A is semiprimitive or JK-semisimple if JK(A) = 0.

5. Establish the Theorem. An alternative algebra A is semiprimitive iff it is a subdirect sum of primitive associative algebras and Cayley algebras over fields.

6. Show that if $A(1 - y)$ generates a proper left ideal $C < A$ then C is contained in a maximal modular left ideal B_y which excludes y. Conclude that if $x \in \bigcap B$ then all elements $y \in I(x)$ have the property that $A(1 - y)$ generates all of A as left ideal. (If y is quasi-invertible then $A(1 - y) = A$ already).

7. If $x \notin B$ for some maximal modular B with modulus e show $y + b = e$ for some $y \in I(x)$, $b \in B$; show $A(1 - y) \subset B$; conclude that if $x \notin B$ some $y \in I(x)$ has the property that $A(1 - y)$ generates a proper left ideal.

8. Establish the Theorem $x \in \bigcap B$ iff $A(1 - y)$ generates A as left ideal for all $y \in I(x)$.

Note that 6-8 go through in any nonassociative algebra.

9. Theorem. The Jacobson-Kleinfeld radical of an alternative algebra contains the Jacobson-Smiley radical,

$$JK(A) \supset Rad(A).$$

10. Theorem If A has dcc. on quadratic ideals then the Jacobson-Kleinfeld and Jacobson-Smiley radicals coincide,

$$JK(A) = Rad(A).$$

#AII.7 Problem Set on Semiprime Centers

We want to show that a semiprime alternative algebra of characteristic $\neq 3$ has a center.

1. If A is semiprime show $\text{Im } 3 \cap \text{Ker } 3 = 0$. Show $\bar{A} = A/\text{Ker } 3$ is nonzero iff $3A \neq 0$; show $\bar{A}$ is still semiprime but has no 3-torsion, and imbeds in a semiprime Ω-algebra $\tilde{A}$ for some ring of scalars with $1/3 \in \Omega$. Conclude $N(\tilde{A}) \neq 0$ and $N(\bar{A}) \neq 0$ and $N(A) \neq 0$ if $3A \neq 0$.
2. Use Problem Sets 00 and 00 to show that either $3A(A) = 0$ or $C(A(A)) = N(A(A)) \neq 0$.
3. Deduce <u>Theorem</u> If A is a semiprime alternative algebra then either $3A = 0$ or $N(A) \neq 0$, and either $3A \subset N(A)$ or $C(A) \neq 0$.
4. Prove <u>Proposition</u> If B is a one-sided ideal in A and A is B-semiprime then either $3B = 0$ or $B \cap N(A) \neq 0$, and either $3B \subset N(A)$ or $B \cap C(A) \neq 0$.
5. Deduce as corollary that if $N(A)$ is a field either $3B = 0$ or $B = A$, and that if $C(A)$ is a field then either $3B \subset N(A)$ or $B = A$.

#AII.8 Problem Set on Weakly Prime Algebras

An algebra is said to be weakly prime if it is semiprime, purely alternative, and faithful as a module over its center Φ (in the sense that $\alpha x = 0$ for $\alpha \in \Phi$, $x \in A$ forces $\alpha = 0$ or $x = 0$).

1. Show that any prime algebra which is not associative is weakly prime. Show that the center Φ of a weakly prime algebra is zero or an integral domain.
2. Show that an algebra with center $\Phi \neq 0$ is weakly prime iff it is a Φ-order in a weakly prime algebra over a field. Show a central algebra over a field is weakly prime iff it is semiprime purely alternative.
3. Show that a weakly prime algebra over a field of characteristic $\neq 3$ is simple. (Use Problem Sets AII.7, 000 and 000).
4. Show that if A is weakly prime and $3A \neq 0$ then A has center $\Phi \neq 0$.
5. Deduce Slater's Weakly Prime Theorem. A weakly prime algebra with $3A \neq 0$ is an order in a Cayley algebra over a field.
6. Show that a prime algebra is either associative or weakly prime.
7. Deduce Slater's Prime Theorem. A prime algebra with $3A \neq 0$ is either associative or an order in a Cayley algebra.

This method of proving Slater's Prime Theorem reduces prime algebras directly to simple algebras; the basic idea is that an ideal has nonzero nucleus = center, hence hits the nucleus = center of the original, and therefore essentially contains an invertible element

8. Prove <u>Proposition</u> If A is semiprime with center $C(A) = \Phi$ a field of characteristic $\neq 3$, then A is associative or a Cayley algebra over Φ.

At first glance this looks like a much more general theorem since one thinks of semiprime algebras as being direct (really subdirect) sums of prime algebras. However, a direct sum decomposition of A would lead to a decomposition of its center, so the condition that the center be a field prevents there being more than one direct summand, so A looks prime.

Problem Set on the Hereditary Nature of the Jacobson-Kleinfeld Radical

1. Show that A is <u>semiprimitive</u> (or JK-<u>semisimple</u>) iff it is a subdirect sum of Cayley algebras over fields and primitive associative algebras. Conclude $JK(A) \supset \mathrm{Rad}\ A$.

2. Show that if A is semiprimitive, so is any ideal $B \triangleleft A$. Conclude $JK(B) \subset B \cap JK(A)$.

3. Use the Minimal Ideal Theorem V.1.11 to show that if A has d.c.c. on ideals then $JK(A) = \mathrm{Rad}\ A$, and therefore $JK(B) = B \cap JK(A)$ for $B \triangleleft A$ in this case.

4. Recall that A is JK-radical iff it has no proper modular left ideals. Show that if $A \xrightarrow{F} \tilde{A}$ where $\tilde{B}$ is a proper modular left ideal in $\tilde{A}$, then $F^{-1}(\tilde{B}) = B$ is a proper modular left ideal in A. Conclude that if A is JK-radical, so is any homomorphic image. If B is a proper modular left ideal in A and $A \xrightarrow{F} \tilde{A}$ an epimorphism with $B \supset \mathrm{Ker}\ F$, show $F(B) = \tilde{B}$ is a proper modular left ideal in $\tilde{A}$.

5. If $B \triangleleft A$ and C is a left B-ideal, show $C_a = C + aC$ is a left B-ideal with $P^+(C_a) \subset C$. If C is a maximal modular left B-ideal, show it is a left A-ideal. Show each maximal modular $C \triangleleft_\ell B$ can be extended to a maximal modular $\tilde{C} \triangleleft_\ell A$ with $\tilde{C} \cap B = C$. Conclude that if A is JK radical, so is any ideal $B \triangleleft A$, and $B \cap JK(A) \subset JK(B)$.

6. Prove the <u>Theorem</u> The Jacobson-Kleinfeld radical is hereditary: for all ideals $B \triangleleft A$ we have

$$JK(B) = B \cap JK(A) .$$