§5. Central and nuclear involutions

We know ~~In Part I we learned~~ quite a lot about the structure of involutions on associative algebras. The easiest involutions on alternative algebras are those whose symmetric part, at least, is associative. We consider involutions where all norms and traces are central or nuclear.

Space (as with other sections, bar after summary + before index of chapters)

In Chapter XII we called an involution on a unital Φ-algebra a scalar involution if all norms $n(x) = xx^*$ were scalars (in the sense that $n(x) \in \Phi 1$). In this cases the traces $t(x) = x + x^*$ are also scalars, since traces $t(x) = n(x + 1) - n(x) - n(1) = n(x,1)$ can be obtained from norms.

We say $*$ is a central involution if all norms and traces lie in the center. All scalar involutions are automatically central. Conversely, all central involutions can be made into scalar involutions:

5.1 (Proposition) If $*$ is a central involution on a Φ-algebra A then $*$ is a scalar involution on A as algebra over its $*$-center Ω.

Proof. Recall that the $*$-center consists of the symmetric elements $\Omega = \mathrm{Sym}(A,*) \cap C(A)$ of the center. Since all traces and norms are central by hypothesis and automatically symmetric, they lie in Ω. Furthermore, $*$ is Ω-linear since $(\omega x)^* = x^*\omega^* = x^*\omega = \omega x^*$. Thus $*$ is an involution on the Ω-algebra A. ∎

In Chapter III we saw that a scalar involution * forces A to have the structure of a degree 2 algebra (with nondegeneracy conditions, even a composition algebra) with * as standard involution.

A more general kind of involution on an alternative algebra is a nuclear involution, defined as one for which all norms $n(x) = xx^*$ and traces $t(x) = x + x^*$ lie in the nucleus. Once more, if A is unital the nuclearity of norms forces nuclearity of traces $t(x) = n(x,1)$. Conversely, under certain conditions the norms follow from the traces. For example, if $\frac{1}{2} \in \Phi$ then all norms and even all symmetric elements are traces since

(5.2) $\quad 2x = x + x^* = t(x) \quad$ if $\quad x^* = x$, $n(y) = \frac{1}{2}t(n(y))$.

But even in characteristic 2 situations

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5.3 (Traces-Imply-Norms Lemma) If all traces $t(x) = x + x^*$ belong to the nucleus of the alternative algebra A, so do all norms $n(x) = xx^*$.

If all traces $t(x) = x + x^*$ lie in the center then so do all norms $n(x) = xx^*$. If all traces are scalars, $t(x) \in \Phi 1$, and if there is at least one invertible trace $t(y)$, then all norms are scalars, $n(x) \in \Phi 1$.

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(when all $x+x^*$ do)

Proof. To show xx^* associates with all y,z in A we compute the associator $[xx^*,y,z] = -[zx^*,y,x] + [x,x^*,y]z + [z,x^*,y]x$ (linearized right bumping) $= +[xz^*,y,x] - [x,x,y] + [z^*,x,y]x$ (because of the basic property $[a^*,b,c] = -[a,b,c]$ if $a^* + a \in N(A)$)

¶ We assume all traces are central. By the foregoing all norms are [illegible] nuclear. We have $n(x) = xx^* = x^*x$ since $x^* = t(x)-x$ commutes with x if $t(x)$ lies in the center. The linearization $n(x,y) = xy^* + yx^* = x^*y + y^*x$ of the norm is already a trace $t(xy^*) = t(x^*y)$. Thus $n(x)y = xx^*y = x\{t(x^*y)-y^*x\} = t(x^*y)x - xy^*x = \{t(xy^*)-xy^*\}x - yx^*x = yn(x)$ by centrality of traces (note that we [illegible] since x, y, x^*, y^* lie in the [illegible] subalgebra generated by x, y and [illegible]). Thus the norms also commute with A and are consequently central.

If traces are scalars then $n(x)t(y) = n(x)y + n(x)y^* = yn(x) + n(x)y^* = t(yn(x))$ is a scalar, so if some $t(y) \in \Phi 1$ is invertible then all norms $n(x) = t(y)^{-1}\, t(yn(x)) \in \Phi 1$ are scalars. ∎

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5.4 (Corollary) If all traces $t(x)$ are scalars in $\Phi 1$ then $*$ is a scalar involution on A as algebra over its $*$-center. ∎

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5.5 (Corollary) If A is a ~~an~~ central algebra over a field Φ with involution such that all traces are scalars, then either $*$ is a scalar involution, or else A is commutative associative of characteristic 2 and $*$ is the identity involution.

Proof. By the above, all norms will be scalars and $*$ a scalar involution unless no trace $t(y)$ is invertible; since Φ is a field, the only way this could happen is for all traces to be zero: $t(x) = x + x^* = 0$. Then $x^* = -x$ for all x, $1 = 1^* = -1$ implies $2 = 0$ and A has characteristic 2. But then $x^* = -x = +x$ is the identity involution, which forces A to be commutative: $xy = x^*y^* = (yx)^* = yx$. In characteristic 2 we have $\frac{1}{3} \in \Phi$, so by Commutativity-Implies-Associativity the commutative algebra A

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We denote the subspace spanned by all norms and traces by N_o (if $1 \in A$ we can get by with norms alone). Since each norm and trace is symmetric, $N_o \subset \mathrm{Sym}(A,*)$. The relation (5.2) shows $2\mathrm{Sym}(A,*) \subset t(A)$, so when $\frac{1}{2} \in \Phi$ we have $N_o = \mathrm{Sym}(A,*)$. In general, a subspace D_o of symmetric elements is called ample if it contains all norms and traces and has the closure property $\boxed{xD_o x^* \subset D_o}$ DISPLAY AS (5.6) for all $x \in A$. (If $1 \in A$ we only need require closure and $1 \in D_o$, since then automatically all norms $n(x) = x1x^*$ and traces $t(x) = n(x,1)$ belong to D_o). The inclusions

$$(5.7) \qquad N_o \subset D_o \subset \mathrm{Sym}(A,*)$$

show that when $1/2 \in \Phi$ the only ample subspace is the space $D_o = \mathrm{Sym}(A,*)$ of all symmetric elements. It is only in characteristic 2 type situations that we need consider arbitrary ample subspaces.

I.8 (Ample Subspace Proposition) If A is an alternative algebra with nuclear involution, then the smallest ample subspace of the nucleus is the subspace N_o spanned by all norms (and traces), and the largest is the subspace $N(A) \cap Sym(A,*)$ of all symmetric elements in the nucleus.

Proof. N_o is already ample because $xn(y)x^* = x(yy^*)x^* = (xy)(y^*x^*) = (xy)(xy)^* = n(xy)$ is a norm and $xt(y)x^* = xyx^* + xy^*x^* = t(xyx^*)$ is a trace by Artin's Theorem ($x, y, N(A)$ generate an associative subalgebra containing $x^* = t(x)-x, y^* = t(y)-y$ if $t(x), t(y) \in N(A)$).

To show the set of all symmetric nuclear elements is ample we need only verify $x N(A) x^* \subset N(A)$ for the nucleus (clearly $x \cdot Sym(A,*) \cdot x^* \subset Sym(A,*)$). By an argument similar to one above, $[(xn)x^*, y, z] = -[zx^*, y, xn] + [xn, x^*, y]z + [z, x^*, y](xn)$ (right bumping) $= +[xz^*, y, xn] - [xn, x, y]z + [z^*, x, y]xn$ (again using $[a^*, b, c] = -[a, b, c]$ by nuclearity of traces) $= [xz^*, y, x]n - [x, x, ny]z - [x, z^*, y]xn$ (moving the nuclear element n out of harm's way by ~~III.(1.6)~~ II.1.6) $= \{[xz^*, y, x] - [x, z^*, y]x\}n = 0$ (right bumping).

(Another proof is based on the relation ~~III.(1.8)~~ II.1.8, $[A, N] \subset N$, so $[xnx^*, y, z] = [[x,n]x^* + nxx^*, y, z] = [x,n][x^*, y, z]$ (~~as~~ $nxx^* \in N$ if $*$ is nuclear) $= -[x,n][x,y,z] = 0$ by ~~III.(1.9)~~ II.1.9). ∎

IX.5 Exercises

1.1 Show $[xnx^*,y,z] = 0$ using the derivation properties of $\Delta_{y,z}$.

1.2 Show $[xnx^*,y,z] = 0$ using the derivation properties of Δ_{x,nx^*}.

1.3 If $t(x)$ is nuclear show $[xx^*,y,x] = 0$ for all y. Linearize to show if all traces are nuclear then $[xx^*,y,z] = 0$ and thus norms are nuclear.

1.4 Use a similar argument to show $[xnx^*,y,z] = 0$ assuming all traces are nuclear (noting $(n - n^*)z^* = [n,z^*] - t(zn) + t(z)n$).