§4. Slater's approach to the structure theory

We have seen that well behaved alternative algebras are either associative or related to Cayley algebras. Michael Slater has developed an illuminating approach to the structure theory which shows how this dichotomy arises naturally: an alternative algebra can be broken intrinsically into an associative part and a purely alternative part.

Fundamental to this approach is the relation between the nucleus of an ideal and the nucleus of the whole algebra. The associative part is the maximal nuclear ideal, while the purely alternative part is the part generated by associators. This separation into two parts leads to a quick proof of an alternative Artin-Wedderburn Theorem for algebras with d.c.c. on left ideals (we apply the associative Artin-Wedderburn to the associative part and break the purely alternative part into Cayley algebras).

Nuclear Inheritance

The key to Slater's approach lies in the nucleus. We will establish that for respectable algebras the nucleus of an ideal $B \triangleleft A$ is the part $B \cap N(A)$ inherited from A. One important consequence of this is that an associative ideal is necessarily nuclear: if $B = N(B)$ associates with itself it is forced to associate $B = N(B) \subseteq N(A)$ with all of A. This is indicative of the sharp dichotomy: the associative part of A is very associative, the purely alternative part is not the slightest bit associative.

We will primarily be concerned with semiprime algebras. We can actually replace "global" semiprimeness of A by "local" semiprimeness at B, where A is **B-semiprime** relative to an ideal B if there are no trivial ideals of A contained inside B. (A may contain trivial ideals, but they all miss B).

Note by III.1.8 that if A is B-semiprime there are no Jordan-solvable ideals of A inside B.

4.1 (Local Nuclear Inheritance Theorem) If an alternative algebra A is B-semiprime for an ideal B, then the nucleus of B is

$$N(B) = B \cap N(A).$$

Proof. Inclusion $B \cap N(A) \subset N(B)$ is trivial: if $z \in B$ nucleizes all of A it certainly nucleizes B. The hard part is the reverse inclusion.

Recall (III.1.3) that the middle annihilator $\mathrm{Ann}_M(B) = \{x \in A \mid U_B x = 0\}$ is an ideal whenever B is, so $C = B \cap \mathrm{Ann}_M(B)$ is a Jordan-trivial ideal of A lying inside B ($U_C C \subset U_B(\mathrm{Ann}_M(B)) = 0$), hence $C = 0$ by B-semiprimeness:

$$(*) \qquad U_B z = 0 \text{ for } z \in B \Rightarrow z = 0.$$

We can apply this together with the fact that associator maps are Jordan derivations (I.3.8) to see

$$(**) \quad [x,y,B] = 0 \text{ for } [x,y,A] \subset B \Rightarrow [x,y,A] = 0$$

since under these hypotheses any $z \in [x,y,A]$ belongs to B yet is killed by B, $U_B z \subset U_B[x,y,A] = [x,y, U_B A] - U_{B,[x,y,B]} A = 0$ by $[x,y,B] = 0$, and therefore by (*) $z = 0$. In particular, for $z \in N(B)$ we have $[z,B,B] = 0$ and $[z,B,A] \subset B$, so by (**) $[z,B,A] = 0$; repeating, $[z,A,B] = 0$ and $[z,A,A] \subset B$ (for $z \in B$) imply $[z,A,A] = 0$ by (**). We have shown $z \in N(B) \Rightarrow z \in N(A)$: nuclearity of z in B and resistance to annihilation by U_B force nuclearity of z in A. ∎

We can establish a similar result for the center.

4.2 (Local Central Inheritance Theorem) If an alternative algebra A is B semiprime for an ideal B, then the center of B is

$$C(B) = B \cap C(A).$$

Proof. Once more $B \cap C(A) \subset C(B)$ is obvious, and by the previous result we at least know the central elements of B nucleize A, $C(B) \subset N(B) \subset N(A)$. It remains to show they commute with A. But using our standard argument we see that for any $c \in C(B)$

$$U_B[c,A] = [c, U_B A] - U_{B,[c,B]}A = 0$$

since $[c,B] = 0$ and commutator maps are Jordan derivations by 1.3.7. Applying (*) we see $[c,A] = 0$ and c is central in A. ∎

For globally semiprime algebras we have

4.3 (Nuclear Inheritance Theorem) If A is a semiprime alternative algebra then the nucleus of any ideal $B \triangleleft A$ is

$$N(B) = B \cap N(A). \quad ∎$$

4.4 (Central Inheritance Theorem) If A is a semiprime alternative algebra then the center of any ideal $B \triangleleft A$ is

$$C(B) = B \cap C(A). \quad ∎$$

4.5 Corollary. Any associative ideal B in a semiprime alternative algebra A is nuclear in A. ∎

We say A is **purely alternative** if it has no nuclear ideals; a typical example would be a Cayley algebra (or a direct sum or product of such). By 4.5, a semiprime alternative algebra is purely alternative iff it contains no associative ideals.

For the corresponding local notion, we say A is **B-purely alternative** for an ideal B if there are no nuclear ideals of A lying inside B. For these we can strengthen the Prime Nucleus Theorem to

4.6 (Semiprime Nucleus Theorem) If an alternative algebra A is B-semiprime and B-purely alternative for an ideal $B \triangleleft A$, then the nucleus and center of B coincide:

$$N(B) = C(B).$$

Proof. We must show $N = N(B)$ commutes with B (hence with A), i.e., the space $M = M(B) = [N(B),A]$ vanishes. We will show M vanishes by showing the ideal $\hat{I}(M)$ it generates vanishes, and we will show $\hat{I}(M)$ vanishes by showing it is a nuclear ideal contained in B and therefore vanishes by B-pure alternativity.

Automatically $\hat{I}(M)$ is an ideal contained in B, so we must establish nuclearity. Now by B-semiprimeness 4.1 and II.1.8 $M \subset N$ are both nuclear in A with

$$[M,A] \subset [N,A] = M$$

$$M[A,A,A] = [N,A][A,A,A] = [N,A[A,A,A]]-A[N,[A,A,A]]$$

$$= [N,A[A,A,A]] \subset [N,A] = M.$$

Thus $\hat{I}(M) = \hat{A}M\hat{A} = [\hat{A},M]\hat{A}+(M\hat{A})\hat{A} = M\hat{A}$ by 0.00, so $\hat{I}(M)$ will be nuclear if $[\hat{I}(M),A,A] = [M\hat{A},A,A] = M[A,A,A]$ vanishes. We will show $M[A,A,A]$ vanishes by showing its spanning elements $w = m[x,y,z]$ ($m = [n,a]$ for $n \in N$, $a,x,y,z \in A$) are trivial nuclear elements and therefore vanish by B-semiprimeness. (By 0.00 $\hat{I}(w)$ is trivial if w is, and is contained in B if w is).

We saw above $w \in M[A,A,A] \subset M$ is nuclear, so all that remains is triviality. Using II.1.9 and its linearization we have $wbw = \{m[x,y,z]\}b\{m[x,y,z]\}$ $= m\cdot[x,y,z]\{bm[x,y,z]\}$ (everything takes place in the associative subalgebra generated by $[x,y,z],b$, and the nucleus) $= -m\cdot[x,y,z]\{bm'[a,y,z]\}$ ($m' = [n,x]$) $= m\cdot[x,y,z]\{[m',b][a,y,z] - m'b[a,y,z]\} = m\cdot[x,y,z]\{-[m',y]\cdot[a,b,z]-[n,x]\cdot b[a,y,z]\}$ $= 0$ since $[x,y,z][m',y] = [x,y,z][n,x] = 0$ by II.1.9. Thus w is indeed trivial. ∎

The associative and purely alternative parts

We wish to divide an algebra into a purely associative and a purely alternative part and analyze these parts separately.

The **associator ideal** Ast(A) of any linear algebra A is the ideal generated by all associators $[x,y,z]$. We always have

$$\mathrm{Ast}(A) = \hat{A}[A,A,A] = [A,A,A]\hat{A}.$$

The two expressions coincide by the Associator Identity II.2.7 ($x[y,z,w] \equiv -[x,y,z]w$ modulo $[A,A,A]$). The first of these is a left ideal since $A\{\hat{A}[A,A,A]\} = (A\hat{A})[A,A,A]-[A,\hat{A}[A,A,A]] \subset \hat{A}[A,A,A]$, dually the second is a right ideal, so their common value is a two-sided ideal. Clearly it is generated by the associators. Note A is associative iff $\mathrm{Ast}(A) = 0$.

We next consider the maximal nuclear ideal of A (which, by Zorn, always exists). This can be described elementwise as the set of properly nuclear elements, where $z \in A$ is **properly nuclear** if it is nuclear and all multiples zA stay nuclear:

$$[z,A,A] = [zA,A,A] = 0.$$

4.7 (Proper Nuclearity Lemma) The following are equivalent for a nuclear element $z \in N(A)$ in an alternative algebra A:

(i) z is properly nuclear

(ii) all za for $a \in A$ are nuclear

(iii) all az for $a \in A$ are nuclear

(iv) $z[A,A,A] = 0$

(v) $[A,A,A]z = 0$

(vi) $z \cdot Ast(A) = 0$

(vii) $Ast(A) \cdot z = 0$.

Proof. (i)$\Leftrightarrow$(ii) by definition, (ii)$\Leftrightarrow$(iv) since $[zA,A,A] = z[A,A,A]$ by Nuclear Slipping II.1.6, (iv)$\Leftrightarrow$(vi) since $zA(A) = z\{[A,A,A]\hat{A}\}$ $= \{z[A,A,A]\}\hat{A}$. Dually (iii)$\Leftrightarrow$(v)$\Leftrightarrow$(vii), and (ii)$\Leftrightarrow$(iii) since always $za-az = [z,a] \in [N,A] \subset N$ by II.1.8. ∎

The properly nuclear elements constitute the *nuclear radical*, which for reasons of euphony is called Nurd(A).

4.8 (Nuclear Radical Proposition) The nuclear radical Nurd(A) is the maximal nuclear ideal of the alternative algebra A: it is an ideal contained in the nucleus, and contains all one-sided nuclear ideals of A. A is purely alternative iff Nurd(A) = 0.

Proof. Clearly Nurd(A) is a subspace by linearity of the defining conditions in 4.6. It is closed under multiplication, since if z is properly nuclear any multiple is not only nuclear but properly nuclear: za is properly nuclear since all $(za)b = z(ab) \in zA$ are nuclear if z and zA are nuclear.

It contains all nuclear right ideals B (say) since $z \in B \Rightarrow zA \subseteq B \subseteq N \Rightarrow z$ is properly nuclear. ∎

The associator ideal Ast(A) and nuclear radical Nurd(A) can be thought of as the purely alternative and associative parts of A respectively.

The nuclear radical and associator ideal are orthogonal,

$$\text{(4.9)} \qquad \mathrm{Nurd}(A)\cdot\mathrm{Ast}(A) = \mathrm{Ast}(A)\cdot\mathrm{Nurd}(A) = 0$$
$$\mathrm{Nurd}(A) = \mathrm{Ast}(A)^{\perp} \cap N(A)$$

by 4.7 (v)-(vi). In particular, since two nonzero ideals in a prime algebra cannot annihilate each other, either $\mathrm{Ast}(A) = 0$ or $\mathrm{Nurd}(A) = 0$:

4.10 (Corollary) A prime alternative algebra is either associative or purely alternative. ∎

We say A is **unmixed** if the nuclear radical and associator ideal stay apart,

$$\mathrm{Nurd}(A) \cap \mathrm{Ast}(A) = 0.$$

A typical example would be a direct sum $A = D \boxplus C$ of an associative algebra D and a Cayley algebra C, where $\mathrm{Nurd}(A) = D$ and $\mathrm{Ast}(A) = C$.

4.11 (Unmixed Proposition) A semiprime alternative algebra A is unmixed, and

$$\mathrm{Nurd}(A) = \mathrm{Ast}(A)^{\perp} \quad .$$

Proof. We have $[\mathrm{Ast}(A)^{\perp},A,A]$ contained in Ast(A) by construction and in $\mathrm{Ast}(A)^{\perp}$ since the latter is an ideal. Since $B \cap B^{\perp} = 0$ for any ideal in a semiprime algebra, we have $[\mathrm{Ast}(A)^{\perp},A,A] \subset \mathrm{Ast}(A) \cap \mathrm{Ast}(A)^{\perp} = 0$ and $\mathrm{Ast}(A)^{\perp}$ is nuclear. In this case 4.9 reduces to $\mathrm{Nurd}(A) = \mathrm{Ast}(A)^{\perp}$. Again semiprimeness implies $\mathrm{Ast}(A) \cap \mathrm{Nurd}(A) = \mathrm{Ast}(A) \cap \mathrm{Ast}(A)^{\perp} = 0$, so A is unmixed. ∎

4.12 (Local Nuclear Radical Inheritance Theorem) If an alternative algebra A is B-semiprime for an ideal B, then the nuclear radical of B is

$$\mathrm{Nurd}(B) = B \cap \mathrm{Nurd}(A).$$

Proof. The inclusion $B \cap \mathrm{Nurd}(A) \subset \mathrm{Nurd}(B)$ holds since the left side is an ideal contained in B and nucleizing all of A. Conversely, if $z \in \mathrm{Nurd}(B)$ is properly nuclear in B we claim it is also properly nuclear in A: $zA \subset N(A)$. It suffices to show $zA \subset N(B)$ since by Local Nuclear Inheritance $N(B) \subset N(A)$. But if z is in Nurd(B) then z and Bz lie in $N(B) \subset N(A)$, so $[B,B,zA] = [B,Bz,A]$ (nuclear Slipping II.1.6 since $z \in N(A)$) $= 0$ (since $Bz \subset N(A)$), and zA is nuclear in B as claimed. ∎

4.13 (Nuclear Radical Inheritance Theorem) If A is a semiprime alternative algebra then the nuclear radical of any ideal B is

$$\mathrm{Nurd}(B) = B \cap \mathrm{Nurd}(A). \quad ∎$$

Pure alternativity means vanishing of the nuclear radical. Also recall that semiprimeness is inherited (IV.3.3).

4.14 (Pure Alternativity Inheritance Theorem) If A is a semiprime purely alternative algebra, so is any ideal $B \triangleleft A$. ∎

Recall that A is *B-purely alternative* if no nuclear ideals of A lie inside B, i.e., $B \cap \mathrm{Nurd}(A) = 0$.

4.15 (Local Pure Alternativity Inheritance Theorem) If A is B-semiprime and B-purely alternative for an ideal $B \triangleleft A$, then B is purely alternative. ∎

Application to algebras with d.c.c. on left ideals

A nice example of the power of this approach is its application to algebras with d.c.c. on left ideals (generalizing our results in Chapter VIII on algebras with d.c.c. on all inner ideals).

4.16 (Artin-Wedderburn Theorem) An alternative algebra is semiprime with d.c.c. on left ideals iff

$$A = \mathrm{Ast}(A) \boxplus \mathrm{Nurd}(A)$$

where the purely alternative part Ast(A) is a finite direct sum of Cayley algebras over fields, and the associative part Nurd(A) is a finite direct sum of Artinian associative matrix algebras over division rings.

Proof. Clearly $\mathbb{C}$ and $M_k(\Delta)$ and any finite direct sum thereof has d.c.c. on one-sided ideals and is semiprime.

For the converse, assume at first that A is semiprime with d.c.c. merely on two-sided ideals. We want to show Ast(A) is unital, hence a direct summand. We do this in a roundabout fashion, first showing that a certain

ideal S is unital and then that S = Ast(A). Let S denote the "socle" of the associator ideal, the sum of all those minimal ideals of A which lie in Ast(A). Since these are precisely the irreducible M(A)-submodules of Ast(A), by module theory we know S is a direct sum of irreducibles, and by the d.c.c. on two-sided ideals this sum must be finite: $S = C_1 \boxplus \cdots \boxplus C_n$. By the Minimal Ideal Theorem III.1.10 a minimal ideal is either trivial or simple, and it can't be trivial in a semiprime algebra, so all minimal ideals B are simple. By Kleinfeld's Simple Theorem B is then Cayley or associative. If $B \subset \mathrm{Ast}(A)$ then $B \cap \mathrm{Nurd}(A) = 0$, so $\mathrm{Nurd}(B) = B \cap \mathrm{Nurd}(A) = 0$ by semiprimeness and Nuclear Radical Inheritance. Such B are not associative, and therefore the socle S of Ast(A) is a finite direct sum $C_1 \boxplus \cdots \boxplus C_n$ of Cayley algebras C_i. Since each C_i is unital, so is S. Then by VIII.0.0 $\mathrm{Ast}(A) = S \boxplus S_0^{\perp}$ where $S_0^{\perp} = S^{\perp} \cap \mathrm{Ast}(A)$ is an ideal in all of A. If $S_0^{\perp}$ were nonzero it would contain a minimal ideal B by the d.c.c., then $B \subset \mathrm{Ast}(A)$ would imply $B \subset S$ by definition of the socle, whereas $B \subset S \cap S_0^{\perp} = 0$ is impossible. Therefore $S_0^{\perp} = 0$ and $\mathrm{Ast}(A) = S \boxplus S_0^{\perp} = S$ is unital. By VIII.0.0 again

$$A = \mathrm{Ast}(A) \boxplus \mathrm{Ast}(A)^{\perp} = \mathrm{Ast}(A) \boxplus \mathrm{Nurd}(A)$$

since $\mathrm{Ast}(A)^{\perp} = \mathrm{Nurd}(A)$ by 4.11.

Up to now we have used only the d.c.c. on two-sided ideals. In dealing with the associative part of A we need the full force of the d.c.c. As a direct summand, Nurd(A) inherits semiprimeness and the d.c.c. on one-sided ideals, therefore is a semiprime artinian associative algebra, so by the associative Artin-Wedderburn Theorem is a finite direct sum of matrix algebras. ∎

4.17 Remark. Notice how conceptually clear the proof is. We break the algebra into purely alternative and associative pieces Ast(A) and Nurd(A), describe these, and show A is their direct sum. The only unfortunate aspect is that the method heavily depends on prior classification of all simple algebras. In some sense it is unsatisfactory to have to use simple algebras to classify semisimple algebras, and it is especially unsatisfactory to have to assume the general theory of simple algebras when one is dealing only with the (presumably) more manageable class of algebras with d.c.c. ■

4.18 Corollary. If A is semiprime with d.c.c. on left ideals then A is unital and has d.c.c. and a.c.c. on both left and right ideals. ■

We can also use our results to compare radicals.

4.19 (Radical Equality Theorem) If A is an alternative algebra with d.c.c. on left ideals then the radicals

$$P(A) = S(A) = L(A) = \mathrm{Nil}(A) = \mathrm{Rad}(A) = JK(A) = BM_c(A)$$

all coincide with the maximal nilpotent ideal.

Proof. Always $P \subset S \subset L \subset \mathrm{Nil} \subset \mathrm{Rad} \subset JK \subset BM_c$ (see IV.0.0). Since $A/P(A)$ is semiprime with d.c.c. we have seen it is a direct sum of unital simple algebras, so by definition $P(A) \supset BM_c(A)$. Thus all radicals coincide. By Zhevlakov-Slater Nilpotence IV.3.14 we know $P(A)$ is the maximal nilpotent ideal. ■

4.20 Remark. Again it is unsatisfactory to have to use our knowledge of simple and semisimple algebras to describe radicals. One would expect a simple direct treatment of radicals as in the associative case (or as in the case of alternative algebras with d.c.c. on inner ideals in IV.9.2). The trouble as always is the refractory nature of one-sided ideals. ∎

Prime Inheritance

The method of middle annihilation used in semiprime and nuclear inheritance applies just as well to prime inheritance. Again we can get by with local primeness. We say A is **B-prime** for an ideal $B \lhd A$ if no orthogonal ideals of A lie inside B: if C,D are nonzero ideals of A lying inside B their product CD is also nonzero. Equivalently, if C is a nonzero ideal of A lying in B then $Ann_{R,B}(C) = B \cap Ann_R(C) = 0$ (though perhaps there are parts of its annihilator lying outside B).

4.21 (Local Prime Inheritance Theorem) If an alternative algebra A is B-prime for an ideal B, then B is prime as an algebra.

Proof. If A is B-prime it is certainly B-semiprime, so by Semiprime Inheritance 0.00 we know B is at least semiprime. To show primeness we will show that $Ann_{R,B}(C) = 0$ for any nonzero ideal $C \lhd B$.

We begin by showing that if $C \lhd B$ is an ideal in B then its annihilator $Ann_B(C)$ is an ideal of A even if C isn't,

$$(*) \quad Ann_B(C) \lhd A \quad (C \lhd B \lhd A) \quad .$$

First note that all annihilators $\mathrm{Ann}_{L,B}(C) = \mathrm{Ann}_{R,B}(C) = \mathrm{Ann}_B(C)$ coincide when B is semiprime, since for example $\mathrm{Ann}_{L,B}(C) \subset \mathrm{Ann}_{R,B}(C)$: $C \cdot \mathrm{Ann}_{L,B}(C) \subset C \cap \mathrm{Ann}_L(C)$ is a trivial ideal in B by $\{C \cap \mathrm{Ann}_L(C)\}^2 \subset \mathrm{Ann}_L(C) \cdot C = 0$, hence it must vanish by the semiprimeness of B. By symmetry, for (*) it suffices if $\mathrm{Ann}_B(C)$ is a left ideal of A. This follows from $\mathrm{Ann}_L(C) \triangleleft_L A$, or equivalently

$$[A, \mathrm{Ann}_L(C), C] = 0,$$

which results from 4.1(**) using B-semiprimeness plus $[\mathrm{Ann}_L(C), C, B] = 0$ plus $[\mathrm{Ann}_L(C), C, A] \subset B$.

If $D = \mathrm{Ann}_B(C)$ were nonzero it would by (*) be a nonzero ideal <u>of A</u> contained in B, so by B-primeness $\mathrm{Ann}_{R,B}(D) = 0$. But C is nonzero and contained in $\mathrm{Ann}_{R,B}(D)$ ($DC = 0$ for $D = \mathrm{Ann}_B(C)$), therefore $\mathrm{Ann}_{R,B}(D) \neq 0$ implies $D = 0$. Thus $\mathrm{Ann}_{R,B}(C) = \mathrm{Ann}_B(C) = 0$ for any nonzero ideal $C \triangleleft B$, and B is prime. ∎

In particular, if A is entirely prime so are all its ideals.

4.22 (Global Prime Inheritance Theorem) If A is a prime alternative algebra, so is any ideal $B \triangleleft A$. ∎

IX.4 Exercises

4.1 Show A is associative iff Ast(A) = 0. We say A is **purely exceptional** if Ast(A) = A; show this is equivalent to the condition that A have no nonzero associative homomorphic images.

4.2 If A is purely exceptional, show Nurd(A) = $A^{\perp}$. Show a one-sided ideal B is nuclear in A if B $\cap$ Ast(A) = 0, and if B is nuclear and A unmixed then B $\cap$ (Ast(A) = 0. Conclude that if A is purely alternative, then Ast(A) is the heart of A and hits all nonzero ideals B.

4.3 Show Ast(A) is the smallest ideal B such that A/B is associative. If A is unmixed show Nurd(A/Nurd(A)) = 0, and Nurd(A) is the smallest ideal B such that A/B is purely alternative. For semiprime A conclude A/Ast(A) is the maximal associative image of A and A/Nurd(A) is the maximal purely alternative image of A.

4.4 Show F(Nurd(A)) $\subset$ Nurd(F(A)) for all homomorphisms $A \overset{F}{\to} \tilde{A}$.

4.5 Show [N(A),N(A)] and any [x,N(A)][x,N(A)] for x $\in$ A are contained in Nurd(A). Conclude that if A is purely alternative, its nucleus is commutative. (Actually N(A) not only commutes with itself, but with all of A - see 0.00).

4.6 If B is a nontrivial minimal ideal in an alternative algebra A, show either (i) B is a Cayley algebra over a field and $B \subset \mathrm{Ast}(A)$, or (ii) B is a simple associative algebra and $B \subset \mathrm{Nurd}(A)$. Show that if C is any other ideal of A (e.g., $C = \mathrm{Ast}(A)$) then $B \not\subset C$ [illegible] $\Rightarrow B \subset C$.

4.7 If A is semiprime and (1) $A/\mathrm{Ast}(A)$ has d.c.c. on left ideals, (2) there is an ideal $B \supset \mathrm{Nurd}(A)$, maximal among those missing $\mathrm{Ast}(A)$, such that all ideals of A/B contain minimal ideals and A/B contains no infinite direct sum of ideals (e.g., both will be met if A/B has d.c.c. on two-sided ideals) show A is a finite direct sum of simple Cayley algebras and simple artinian associative matrix algebras.

4.8 Assume (1) A is semiprime, (2) A is purely alternative, (3) every ideal of A contains a minimal ideal, (4) every descending chain $B_1 > B_2 > \cdots$ of direct summands of A eventually terminates. Show A is a finite direct sum of Cayley algebras, in particular is unital with a.c.c. and d.c.c. on left and right ideals. Construct examples of alternative algebras A_k satisfying all of (1)-(4) except (k).

4.9 Assume (1) A is $\mathrm{Ast}(A)$-semiprime, (2) all A-ideals inside $\mathrm{Ast}(A)$ contain minimal A-ideals, (3) if $B_1 > B_2 > \cdots$ is a chain of direct summands of A then the chain $B_1 \cap \mathrm{Ast}(A) \supseteq B_2 \cap \mathrm{Ast}(A) \supseteq \cdots$ terminates. Show that $A = C_1 \boxplus \cdots \boxplus C_n \boxplus D$ where the C_i are Cayley algebras and D is associative.

4.10 If $B \triangleleft A$ where A has d.c.c. on left ideals contained in B and B has the property that its image $F(B)$ has no nuclear idempotents in any image $F(A)$, show B is nilpotent.

4.11 If $B \triangleleft A$ where A is a p.i. algebra (see Appendix) and has d.c.c. on left ideals contained in B, where B has the property that its image $F(B)$ has no central idempotents in any image F(A), show B is nilpotent.

4.12 If A is a strongly semiprime alternative algebra, show that $U_z B = 0 \Rightarrow z \in \mathrm{Ann}\, B$ for any ideal $B \triangleleft A$ (more generally, $z \in \mathrm{Ann}_R(B)$ if B is a left ideal and $z \in \mathrm{Ann}_L(B)$ if B is a right ideal). Recall that A is strongly prime if it is prime and strongly semiprime, i.e.

(*)
$$U_C B = 0 \text{ for } B, C \triangleleft A \text{ implies } C = 0 \text{ or } B = 0.$$
$$U_z A = 0 \text{ implies } z = 0.$$

Conclude that an alternative algebra is strongly prime iff

(**)
$$U_z B = 0 \text{ for } B \triangleleft A \text{ implies } z = 0 \text{ or } B = 0.$$

4.13 Improve on Nuclear Inheritance to show $N(B) = B \cap N(A)$ for a one-sided ideal B in A if merely $\mathrm{Ker}(B) \cap B^{\perp,M} \cap [N(B),A,A] = 0$ (weakening B-semiprimeness). Indeed, show that if $z \in N(B)$ has $\mathrm{Ker}(B) \cap B^{\perp,M} \cap [z,A,A] = 0$ then $z \in N(A)$.

4.14 If B is any one-sided ideal in A, and $b \in B$ satisfies $[b,A,B] = 0$, show $[b,A,A] \subset \mathrm{Ker}(B)$.

4.15 If A is B-semiprime for a left ideal B, show $[\mathrm{Ann}_L(B),B,A] = 0$, and $B \cap \mathrm{Ann}_L(B)$ is a trivial left ideal of A contained in B.

4.16 Show A is B-semiprime for $B \triangleleft A$ iff $C \cap B = 0$ for all trivial ideals $C \triangleleft A$.

4.17 If $A \triangleleft B$ show A is B-semiprime iff B is semiprime as an algebra.

4.18 If B is a left ideal in A, show $\mathrm{Ann}_L(B)$ is an ideal in A iff $[\mathrm{Ann}_L(B),B,A] = 0$. Use middle annihilation to show $[\mathrm{Ann}_L(B),B,A] = 0$ if A is Ker(B)-semiprime and $[\mathrm{Ann}_L(B),B,A] \subset \mathrm{Ker}(B)$. If A is prime and $\mathrm{Ann}_L(B)$ is nonzero, show $B = 0$.

4.19 Show A is left B prime for B a left ideal of A (i.e., if C,D are nonzero left ideals of A contained in B then $CD \neq 0$) iff $CD = 0$ for C,D left ideals of A implies $C \cap B = 0$ or $D \cap B = 0$, iff $CD \cap B = 0$ for C,D left ideals of A implies $C \cap B = 0$ or $D \cap B = 0$.

IX.4.1. Problem Set on One-sided Inheritance

We can carry many of our inheritance results over to one-sided ideals. If B is a left ideal in A we say A is **B-semiprime** if there are no trivial left ideals of A contained in B. (Warning: this does not imply B is semiprime as an algebra, see Ex. 10 below). Recall (III.2.1-3) that the smallest two-sided ideal containing B is its hull $H(B) = B\hat{A}$, the largest two-sided ideal contained in B is its kernel $Ker(B) = \{b \in B \mid bA \subset B\}$, where any associator with one factor from B falls back in B and with two factors from B falls into the kernel: $[A,A,B] \subset B$ and $[A,B,B] \subset Ker(B)$.

1. If B is actually a two-sided ideal, show this notion of one-sided B-semiprimeness coincides with our earlier definition of two-sided B-semiprimeness. If $C \subset B$ are left ideals show B-semiprimeness implies C-semiprimeness.
2. If B is a left ideal and A is Ker(B)-semiprime, use (4.1**) to show $[N(B),B,A] = 0$; if A is B-semiprime use it to show further $[N(B),A,A] = 0$.
3. Deduce the Local One-sided Nuclear Inheritance Theorem: If an alternative algebra A is B-semiprime for some one-sided ideal B then the nucleus of B is

$$N(B) = B \cap N(A).$$

4. Mimic the two-sided proof to establish the Local One-sided Nuclear Radical Inheritance Theorem: If an alternative algebra A is B-semiprime for some one-sided ideal B then

$$Nurd(B) = B \cap Nurd(A).$$

5. Examine the proofs of Ex. 3.4 in detail and show that $N(B) = B \cap N(A)$, $Nurd(B) = B \cap Nurd(A)$ hold if we merely assume A is C-semiprime for $C = I_L([N(B),A,A]) \subset B$. [Hint: for Ex. 4 the relevant ideal is $D = I_L([ANurd(B),A,A])$; show $D \subset C$ by showing $ANurd(B) \subset Nurd(B) \subset N(B)$ via the usual application of (4.1**); note $[ANurd(B),B,B] = [BNurd(B),B,A]$ by right bumping since $Nurd(B) \subset N(B) \subset N(A)$ using Ex. 3.]

6. For the center we need local semiprimeness of the hull H(B) rather than just B itself, since $[C(B),A]$ needn't be contained in B any longer if B is merely one-sided. Show $[C(B),A]H(B) = 0$ if A is B-semiprime. If $H(B) \cap Ann_L(H(B)) = 0$ show $[C(B),A] = 0$.

7. Deduce the Local One-sided Central Inheritance Theorem: If an alternative algebra A is H(B)-semiprime for some one-sided ideal B, then the center of B is

$$C(B) = B \cap C(A).$$

8. Establish the Global One-sided Inheritance Theorem: If A is a semiprime alternative algebra then for all one-sided ideals B

$$N(B) = B \cap N(A)$$
$$C(B) = B \cap C(A)$$
$$Nurd(B) = B \cap Nurd(A).$$

9. If A is B-semiprime and B-purely alternative for a one-sided ideal B in A (in the sense that no one-sided nuclear ideal of A lies inside B, $B \cap Nurd(A) = 0$) show B is purely alternative.

10. It is not in general true that a one-sided ideal in a semiprime algebra is semiprime; give a simple counterexample. However, such counterexamples exist only in the presence of associativity: in the purely alternative

case semiprimeness is inherited. If A is B-semiprime show $C \cap \mathrm{Ker}(B) = 0$ for any trivial ideal $C \lhd B$ by Ker(B)-semiprimeness, then show C is nuclear in B, and use this to deduce the <u>One-sided Semiprime Inheritance Theorem</u>: If an alternative algebra A is B-semiprime and B-purely alternative for a one-sided ideal B, then B is semiprime and purely alternative as an algebra.

11. If $BC = 0$ for left ideals B,C of A show $H(B)H(C) = 0$. Derive the <u>One-sided Primeness Theorem</u>: If A is a prime alternative algebra then the product BC of two nonzero left (or right) ideals B,C is again nonzero.

If B is a one-sided ideal in an associative algebra A, there is no reason why B should contain a nonzero ideal of A (e.g., $B = M_n(\Phi)$ simple, $B = Ae_{11}$). However, if A is a Cayley algebra its only one-sided ideals are $B = 0$ and $B = A$, so every nonzero one-sided ideal <u>is</u> a nonzero two-sided ideal. This holds more generally.

12. Show that if A is B-semiprime for a nonzero left ideal B then $\mathrm{Ker}(B) = 0$ implies $B \subset \mathrm{Nurd}(A)$. Derive the <u>Proposition</u>: If an alternative algebra A is B-semiprime and B-purely alternative for some nonzero one-sided ideal B then B contains a nonzero two-sided ideal of A. In particular, if A is semiprime and purely alternative then every nonzero one-sided ideal contains a nonzero two-sided ideal.

Thus one-sided ideals are not far from being two-sided in the purely alternative case.

IX.4.2 Problem Set on Nuclear Inheritance

We outline an alternate approach to nuclear inheritance which avoids Jordan methods.

1. Show that if $B \triangleleft A$ and $n \in N(B)$, $a \in A$, $b,c \in B$ then $c[n,b,a] = [n,bc,a] = [c,[b,n],a] = -[n,b,a]c$ is an alternating function of b,c.

2. If $z = [n,b,a]$ as above, show $z^2 = z \circ B = U_z B = 0$ so $\hat{B}z = z\hat{B}$ is a trivial ideal of B. Conclude that if B is semiprime then $[N(B),B,A] = 0$.

3. Alternately, for $n \in N(B)$ and $a \in A$ show $R = A_{n,a}(B)$ and $K = \text{Ker } A_{n,a}$ are ideals in B with $KR = 0$, so $K \cap R \triangleleft B$ is trivial. If $\text{Ann}_{L,B}(B) = 0$ show $K = \text{Ann}_L(R)$. Show that r^2, $r \circ B$, $U_r B$, $U_B r$, $U_{r,B} B$ all lie in $K \cap R$ for $r \in R$. Deduce that if B is semiprime then $R = 0$ and $[n,a,B] = 0$.

One way to be assured B is semiprime is to assume A is B-semiprime (using Semiprime Inheritance IV.3.3). We can avoid Semiprime Inheritance by the following detour.

4. For $z = [n,b,a]$ as above show $z[B,B,B] = 0$, $z(U_B B) = 0$, $z(Bc^2) = 0$. Show $2BB^2 \subset [B,B,B] + U_B\hat{B} + B \cdot U_B 1$ and conclude $2\{(zB)B\}B = 0$. If $\text{Ann}_{L,B}(B) = 0$ (as when A is B-semiprime) conclude $2z = [z,B,B] = 0$, and therefore $[z,B] = [z,B,B] = 0$ and $z \in C(B)$. Use this and Exercise 1 to show $[z,B,A]B = 0$, so $\text{Ann}_{L,B}(B) = 0$ implies $[z,B,A] = 0$.

5. Whenever $w \in B \triangleleft A$ satisfies $wB = Bw$ and $[w,B,A] = 0$ show $Bw = wB \triangleleft A$; if w is trivial in B show Bw is trivial; conclude that if A is B-semiprime then $w = 0$. Use this to prove $[N(B),B,A] = 0$ when A is B-semiprime.

Returning to where we took the detour, assume B is an ideal in A and $n \in B$ satisfies $[n,B,A] = 0$.

6. Show $[n,A,A]B = 0$, and conclude that $[n,A,A] = 0$ if $\mathrm{Ann}_{L,B}(B) = 0$. Establish the Nuclear Inheritance Theorem 4.1.

7. If $c \in C(B)$ show $[c,A]B = 0$, so $[c,A] = 0$ if $\mathrm{Ann}_{L,B}(B) = 0$. Establish the Central Inheritance Theorem 4.2.

We can similarly establish the one-sided theorems without Jordan methods. Throughout we assume B is a left ideal, $C = \mathrm{Ker}(B)$ the maximal two-sided ideal inside it.

8. As in Exercise 1, show $c[n,b,a] = [n,bc,a] = [c,[n,b],a] = -[n,b,a]c$ for $a \in A$, $b \in B$, $c \in C$, $n \in N(B)$ vanishes if $b = c \in C$.

9. For $z = [n,b,a]$ show $z \in C$ has $z^2 = z \circ C = U_z C = 0$, so $\hat{C}z = z\hat{C}$ is a trivial ideal of C. Conclude that $[N(B),B,A] = 0$ if A is B-semiprime (using Semiprime Inheritance). An alternate method avoids Semiprime Inheritance: show $C[N(B),C,A] = 0$ using $[N(B),C] \subset N(C) \subset N(A)$, so if A is B-semiprime $[N(B),C,A] = 0$; then show $C[N(B),B,A] = 0$, so $[N(B),B,A] = 0$ in the B-semiprime case (using $[N(B),B,A] \subset C$).

10. When $[N(B),C,A] = 0$ show $[N(B),A,A]C = 0$ as in Exercise 6. If $\mathrm{Ann}_{L,B}(C) = 0$, conclude $N(B) \subset N(A)$.

11. Derive the One-sided Nuclear Inheritance Theorem.

12. If A is B-semiprime for a left ideal B, show $[C(B),A]H(B) = 0$. Deduce $C(B) = B \cap C(A)$ when A is $H(B)$-semiprime (or even just $I([C(B),A])$-semiprime).

IX.4.3 Problem Set on Associative and Pure Alternative Parts

Although the associator ideal and nuclear radical are ideals in the original algebra which usually form a direct sum, $A \supset Ast(A) \boxplus Nurd(A)$, they don't always add up to the whole algebra. Another way to break A into associative and purely alternative pieces is by means of quotients: define the **associative part** of A to be A/Ast(A), and the **purely alternative part** to be A/Nurd(A) (despite the fact that these are homomorphic images rather than subalgebras of A).

1. Show A/Ast(A) is the maximal associative image of A. Despite its name, A/Nurd(A) is not always purely alternative (give an example where A is solvable of index 2); show, however, that it is at least maximal in the sense that if A/B is purely alternative then $B \supset Nurd(A)$).

2. If A is unmixed show A/Nurd(A) is purely alternative. If A is semiprime show A/Nurd(A) contains no associative ideals, in particular is semiprime.

3. Prove the <u>Unmixed Theorem</u>. An unmixed alternative algebra is a subdirect sum

$$A \lessapprox A/Ast(A) \boxplus A/Nurd(A)$$

of its associative and purely alternative parts (which are the maximal associative and purely alternative images of A respectively).

This is the natural or intrinsic decomposition of a semiprime algebra into its associative and purely alternative parts.

4. Use 3.8 and 4.10 to obtain a non-intrinsic decomposition of a semiprime algebra into a semidirect sum of an associative and a purely alternative algebra. (Compare with 3.9).

These notions of associative and purely alternative parts can be used to re-derive the Artin-Wedderburn Theorem 4.16. Their advantage is that a quotient A/B automatically inherits d.c.c., whereas we had to resort to subterfuge to assure ourselves that the ideals Nurd(A) and Ast(A) inherited the d.c.c.

5. If A is semiprime with d.c.c. on left ideals, and B is a maximal ideal of A containing Ast(A) but missing Nurd(A) [remark: we can show Nurd(A) is maximal among ideals missing Ast(A), but it is not clear whether Ast(A) is maximal among ideals missing Nurd(A)], show A/B is a semiprime associative algebra with d.c.c. on left ideals; show Nurd(A) is imbedded as an ideal in A/B, and deduce Nurd(A) is a finite direct sum of simple artinian matrix algebras, so in particular Nurd(A) is unital.

6. Under the same hypotheses show the purely alternative part A/Nurd(A) is semiprime with d.c.c. Show it is a finite direct sum of Cayley algebras (using the Minimal Ideal Theorem and Kleinfeld's Simple Theorem). Deduce that Ast(A) is a finite direct sum of Cayley algebras, in particular is unital.

7. Derive the Artin-Wedderburn Theorem by showing that a minimal ideal in a semiprime algebra A belongs either to Ast(A) or Nurd(A), so that if Ast(A) $\boxplus$ Nurd(A) is unital its complement contains no minimal ideals of A.

IX.4.4 Problem Set on Nucleus and Center

How far the nucleus is from being central is measured by the ideal Curd(A) generated by all commutators $[a,n]$ for $a \in A$ and $n \in N(A)$; Curd(A) = 0 iff all $[a,n] = 0$ iff $N(A) = C(A)$.

1. Show Curd(A) = $\hat{A}[A,N(A)] = [A,N(A)]\hat{A}$.

2. Show Curd(A) $\subset$ Nurd(A) $\Longleftrightarrow$ Curd(A)$\cdot$Ast(A) = 0 $\Longleftrightarrow$ $[A,N(A)]\cdot[A,A,A] = 0$ $\Longleftrightarrow$ [Curd(A),A,A] = 0. Usually Curd(A) will be contained in Nurd(A). Show that although [Curd(A),A,A] may not always be zero, at least it is always contained in N(A).

3. Show any $m = [a[x,n],b,c]$ (for $a,b,c,x \in A$, $n \in N(A)$) is a trivial element of the nucleus. Conclude that either Curd(A) $\subset$ Nurd(A) or else there is a trivial ideal I(m) where m is contained in Curd(A) $\cap$ Ast(A) $\cap$ N(A). (We do not claim all of I(m) is contained in N(A); see SN 2 in the next problem set).

4. Deduce <u>Slater's General Nuclear Theorem</u>: If Curd(A) $\cap$ Ast(A) $\cap$ Nurd(A) contains no trivial elements then Curd(A) $\subset$ Nurd(A), so that if A is also purely alternative then N(A) = C(A).

5. Deduce <u>Slater's Nuclear Theorem</u>: If A is semiprime then Curd(A) $\subset$ Nurd(A). If A is semiprime and purely alternative then its nucleus and center coincide, N(A) = C(A).

6. Deduce N(A) = C(A) also if A has no associative ideals, or is semiprime with no associative images (= purely exceptional).

7. If A has no trivial nuclear elements show: (i) A is unmixed, (ii) Curd(A) $\subset$ Nurd(A), (iii) N(A) = C(A) if A is purely alternative, (iv) Ast(A) $\cap$ N(A) = Ast(A) $\cap$ C(A), (v) [N(A),Ast(A)] = 0.

IX.4.5 Problem Set on Slater's Nuclear Conjectures

Michael Slater has made the following conjectures about the distance of the nucleus from the center in the purely alternative case:

(SN 1) Curd(A) $\cap$ Ast(A) is zero or contains trivial nuclear ideals

(SN 2) Curd(A) $\subset$ N(A) or else Curd(A) $\cap$ Ast(A) contains trivial nuclear ideals

(SN 3) A unmixed $\Rightarrow$ Curd(A) $\subset$ N(A)

(SN 4) If Nurd(A) is commutative without nilpotent elements, then N(A) = C(A).

(SN 5) A purely alternative $\Rightarrow$ N(A) = C(A) .

1. Show in general $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Leftrightarrow 5$ for alternative algebras. In the previous problem set we saw (SN 4) always holds.

2. Let $g(x_i,y_i,z_i;w_j;n_j) = \Pi_{i=1}^{r}[x_i,y_i,z_i]\Pi_{j=1}^{s}[w_j,n_j]$ $(r,s \geq 1)$ for $x_i,y_i,z_i,w_j \in A$ and $n_j \in N(A)$. Show g is independent of the order and association of the factors $[x_i,y_i,z_i]$ and $[w_j,n_j]$. Show its values lie in Curd(A) $\cap$ N(A).

3. Show g is an alternating function of its arguments x_i,y_i,z_i,w_j, which vanishes if one of these variables lies in N(A). If $w_s = w_s'w_s''$ show $g(x,y,z;w_s;n) = g(x,y,z;w_s';n)w_s'' + g(x,y,z;w_s'';n)w_s' + g(x,y,z;w_s';n_s'')$ where $n_s'' = [w_s'',n_s]$. Conclude that if g vanishes on a set of generators x,y,z,w for A modulo N(A), it vanishes everywhere.

4. Conclude that if A is finitely generated mod N(A) then g = 0 for large enough r and s. Show SN 2-5 hold if A is finitely generated mod N(A).

5. If A is generated mod N(A) by 3 elements, show Curd(A) $\subset$ N(A).

IX.4.6 Problem Set on Prime Inheritance

We give an alternate proof of prime inheritance, based on the nuclear radical rather than Jordan methods.

1. If A is B-prime for an ideal B (or **left B-prime** for a left ideal B in the sense that there are no orthogonal left ideals of A inside B) show either $B \cap Ast(A) = 0$ or $B \cap Nurd(A) = 0$; conclude either B is properly nuclear or B is semiprime purely alternative without one-sided associative ideals.

2. If C,D are orthogonal left ideals of A contained in an ideal B, show CB,DB are orthogonal ideals of A contained in B if B is properly nuclear; if A is B-prime, conclude C = 0 or D = 0.

3. If C,D are orthogonal left ideals of A contained in an ideal B, show Ker(C), Ker(D) are orthogonal ideals of A contained in B; if A is B-prime conclude Ker(C) = 0 or Ker(D) = 0. If B is semiprime purely alternative, deduce C = 0 or D = 0.

4. Prove the <u>Theorem</u>. If A is B-prime for an ideal B then it is left and right B-prime.

5. Deduce the One-sided Primeness Theorem.

6. Use the associator derivation formula 0.00 to show $[A,C,Ann(C)]B = 0$ if $C \lhd B \lhd A$; if A is B-prime conclude $Ann_B(C) = Ann_{L,B}(C) = Ann_{R,B}(C) \lhd A$. Deduce the Prime Inheritance Theorem.

As with semiprimeness, primeness will not in general be inherited by a mere one-sided ideal (same example: $A = M_n(\Phi)$ is simple associative, hence prime, but the left ideal $B = Ae_{11}$ is not even semiprime, much less prime). Under certain conditions primeness will be inherited.

7. If B is a left ideal of A, where A is left-B-prime and $B \not\subset$ Nurd(A), show B is prime as an algebra. (Use Exercise 1 and 0.00).

8. If $B \subset$ Nurd(A) is a left ideal of A, where A is left-B-prime and $Ann_{L,B}(B) = 0$, show B is prime as an algebra. (Copy Exercise 2).

In addition to primeness, weak primeness is inherited. We say A is **B-weakly prime** for an ideal $B \triangleleft A$ if A is B-semiprime and B-purely alternative, and if in addition A is **B-torsion-free** (the elements β of $B \cap C(A) = C(B)$ act injectively on A: $\beta a = 0 \Rightarrow \beta = 0$ or $a = 0$).

9. A itself is **weakly prime** if it is A-weakly prime. Show A is B-weakly prime iff B is weakly prime as an algebra.

IX.4.7 Problem Set on Minimal Left Ideals

We wish to obtain for one-sided ideals an analogue of the Minimal Ideal Theorem. We will make free use of nuclear inheritance.

1. If B is a minimal one-sided ideal in A, show the ideal generated by $[B,B,A]$ is either 0 or B. If $\hat{I}([B,B,A]) = B$ deduce B is actually minimal two-sided ideal of A contained in Ast(A). In this case show B is a simple Cayley algebra, in particular is unital: $B = Ae = eA$ for a central idempotent e.

2. If $\hat{I}([B,B,A]) = 0$ show B is associative; if B is not trivial show A is B-semiprime, and use Local One-sided Nuclear Inheritance (Problem Set 4.6) to conclude $B \subset \mathrm{Nurd}(A)$.

3. If B is a non-trivial minimal left ideal of A contained in Nurd(A), show $\hat{I}(B) = B\hat{A} = B\hat{I}(B) \subset \mathrm{Nurd}(A)$, and that B is a minimal left ideal of the algebra $\hat{I}(B)$. Conclude that $B = \hat{I}(B)e = Ae$ for a nuclear idempotent e.

4. If B is as in Ex. 3, and if $\mathrm{Ann}_L(\hat{I}(B)) = 0$, show $\hat{I}(B)$ is simple. Show the annihilator condition is necessary by constructing a 2-dimensional example with B a field and $\hat{I}(B) = A$ not semiprime.

5. Prove the Minimal Left Ideal Theorem. A minimal left ideal B of an alternative algebra A is either

   (i) trivial, $B^2 = 0$;

   (ii) a simple two-sided ideal contained in Ast(A) and isomorphic to a Cayley algebra, $B = \hat{I}(B) \subset \mathrm{Ast}(A)$;

   (iii) a minimal left ideal of a nuclear ideal $\hat{I}(B)$, B minimal in $\hat{I}(B) = B\hat{I}(B) \subset \mathrm{Nurd}(A)$.

In cases (ii)-(iii) B is of the form B = Ae for a nuclear idempotent e.

6. Prove Hopkins' Theorem. If an alternative algebra A has d.c.c. on left ideals, then any non-nilpotent left ideal B contains a nonzero idempotent e. [Hint: A/P(A) is semiprime with d.c.c., $\bar{B}$ a nonzero left ideal, hence contains a minimal left ideal $\overline{Ae}$; we can lift $\bar{e}$ in $\bar{B} \cong B/B \cap P(A)$ through the nil ideal P(A) to an idempotent $e \in B$].

7. Show directly that the Jacobson-Kleinfeld radical JK(A) and Jacobson-Smiley radical Rad(A) coincide for alternative algebras with d.c.c. on left ideals. [Hint: Assume Rad(A) = 0, so by semiprimeness a minimal left ideal in JK(A) gives rise to a nuclear idempotent $e \in JK(A)$.]

It has been conjectured that if A is purely alternative with d.c.c. on left ideals then A has d.c.c. on right ideals. (By 4.18 this is known if A is semiprime).

IX.4.8 Problem Set on Goldie Theory

Here we develope an analogue of the Goldie theory for associative algebras with a.c.c. An arbitrary linear algebra A is a **left order** in $\tilde{A}$ (or $\tilde{A}$ is a classical algebra of **left fractions** or a **left quotient algebra** of A) if $A \subset \tilde{A}$ and

(1) all nonsingular nuclear elements of A are invertible and nuclear in $\tilde{A}$,

(2) all elements of $\tilde{A}$ may be expressed as left fractions $x^{-1}y$ for $x,y \in A$ with x nonsingular and nuclear.

An element x is **nonsingular** if it is not a zero divisor, xy = 0 or yx = 0 implies y = 0, i.e., L_x and R_x are nonsingular linear transformations.

1. Show A has an algebra of left fractions iff for all elements (resp. nonsingular nuclear elements) $x \in A$ and nonsingular nuclear $y \in A$ there exist $x' \in A$ (resp. nonsingular nuclear x') and nonsingular nuclear $y' \in A$ with $y'\cdot x = x'\cdot y$. If A is alternative, show x,y,y' nuclear and nonsingular imply such an element x' is automatically nuclear and right nonsingular ($ax' = 0 \Rightarrow a = 0$); show it is left nonsingular ($x'a = 0 \Rightarrow a = 0$). [Hint: there exists a nonsingular nuclear y'' and an x'' with $y''\cdot y = x''\cdot y'x$ since y'x is nonsingular; show y''a = 0].

2. Show a classical algebra of left fractions is unique up to A-isomorphism (if it exists), i.e., if $A \subset \tilde{A}_1$ and $A \subset \tilde{A}_2$ are algebras of left fractions there is an isomorphism $\tilde{A}_1 \to \tilde{A}_2$ which is the identity on A.

3. Prove the Theorem. If A is a prime purely alternative algebra of characteristic $\neq 3$ then A has an algebra of left fractions which is simple with d.c.c. on left ideals (in fact, is a Cayley algebra over its center with no proper left ideals at all.)

Note that, unlike the associative case, we need assume no chain conditions on A at all.

4. Deduce from the previous theorem that if A is prime purely alternative, any nonzero left ideal B has zero left and right annihilators. Prove this directly by establishing the Lemma: If B is a nonzero left ideal in A where A is B-semiprime and B-purely alternative (see 0.00 and 0.00), then B contains a nonzero ideal of A.

5. Formulate and prove a Goldie Theorem for abitrary prime alternative algebras of characteristic $\neq 3$. What can you do for semiprime purely alternative algebras?

IX.4.9 Problem Set on Semiprime Centers

We want to show that a semiprime alternative algebra of characteristic $\neq 3$ has a center.

1. If A is semiprime show Im $3 \cap$ Ker $3 = 0$. Show $\bar{A} = A/\text{Ker } 3$ is nonzero iff $3A \neq 0$; show $\bar{A}$ is still semiprime but has no 3-torsion, and imbeds in a semiprime Ω-algebra $\tilde{A}$ for some ring of scalars with $1/3 \in \Omega$. Conclude $N(\tilde{A}) \neq 0$ and $N(\bar{A}) \neq 0$ and $N(A) \neq 0$ if $3A \neq 0$.

2. Use Problem Sets 00 and 00 to show that either $3\,\text{Ast}(A) = 0$ or $C(\text{Ast}(A)) = N(\text{Ast}(A)) \neq 0$.

3. Deduce <u>Theorem</u>: If A is a semiprime alternative algebra then either $3A = 0$ or $N(A) \neq 0$, and either $3A \subset N(A)$ or $C(A) \neq 0$.

4. Prove <u>Proposition</u>: If B is a one-sided ideal in A and A is B-semiprime then either $3B = 0$ or $B \cap N(A) \neq 0$, and either $3B \subset N(A)$ or $B \cap C(A) \neq 0$.

5. Deduce as corollary that if $N(A)$ is a field either $3B = 0$ or $B = A$, and that if $C(A)$ is a field then either $3B \subset N(A)$ or $B = A$.

IX.4.10 Problem Set on Weakly Prime Algebras

An algebra is said to be weakly prime if it is semiprime, purely alternative, and faithful as a module over its center Φ (in the sense that $\alpha x = 0$ for $\alpha \in \Phi$, $x \in A$ forces $\alpha = 0$ or $x = 0$).

1. Show that any prime algebra which is not associative is weakly prime.

 · Show that the center Φ of a weakly prime algebra is zero or an integral domain.

2. Show that an algebra with center $\Phi \neq 0$ is weakly prime iff it is a Φ-order in a weakly prime algebra over a field. Show a central algebra over a field is weakly prime iff it is semiprime purely alternative.

3. Show that a weakly prime algebra over a field of characteristic $\neq 3$ is simple, and therefore a Cayley algebra. (Use Problem Sets IX.2.6,00 and 00).

4. Show that if A is weakly prime and $3A \neq 0$ then A has center $\Phi \neq 0$.

5. Deduce <u>Slater's Weakly Prime Theorem</u>. A weakly prime algebra with $3A \neq 0$ is an order in a Cayley algebra over a field.

6. Show that a prime algebra is either associative or weakly prime.

7. Deduce <u>Slater's Prime Theorem</u>. A prime algebra with $3A \neq 0$ is either associative or an order in a Cayley algebra.

This method of proving Slater's Prime Theorem reduces prime algebras directly to simple algebras; the basic idea is that an ideal has nonzero nucleus = center, hence hits the nucleus = center of the original, and therefore essentially contains an invertible element.

8. Prove Proposition: If A is semiprime with center $C(A) = \Phi$ a field of characteristic $\neq 3$, then A is associative or a Cayley algebra over Φ.

At first glance this looks like a much more general theorem since one thinks of semiprime algebras as being direct (really subdirect) sums of prime algebras. However, a direct sum decomposition of A would lead to a decomposition of its center, so the condition that the center be a field prevents there being more than one direct summand, so A looks prime.