§4. Slater's approach to the structure theory

We have seen that well behaved alternative algebras are either associative or related to Cayley algebras. Michael Slater has developed an illuminating approach to the structure theory which shows how this dichotomy arises naturally: an alternative algebra can be broken intrinsically into an associative part and a purely alternative part.

Fundamental to this approach is the relation between the nucleus of an ideal and the nucleus of the whole algebra. The associative part is the maximal nuclear ideal, while the purely alternative part is the part generated by associators. This separation into two parts leads to a quick proof of an alternative Artin–Wedderburn Theorem for algebras with d.c.c. on left ideals (we apply the associative Artin–Wedderburn to the associative part and break the purely alternative part into Cayley algebras).

Nuclear Inheritance

The key to Zelmanov's approach lies in the nucleus. We will establish that for semiprime algebras the nucleus of an ideal $B \triangleleft A$ is the part $B \cap N(A)$ inherited from A. One important consequence of this is that an associative ideal is necessarily nuclear: if $B = N(B)$ associates with itself it is forced to associate $B = N(B) \subseteq N(A)$ with all of A. This is indicative of the sharp dichotomy: the associative part of A is very associative, the purely alternative part is not the slightest bit associative.

We will primarily be concerned with semiprime algebras. We can actually replace "global" semiprimeness of A by "local" semiprimeness at B, where A is B-semiprime relative to an ideal B if there are no trivial ideals of A contained inside B. (A may contain trivial ideals, but they all miss B). Note by IV, 3, ? that if A is B-semiprime there are no Jordan-solvable ideals of A inside B.

4.1 (Local Nuclear Inheritance Theorem) If an alternative algebra A is B-semiprime for an ideal B, then the nucleus of B is

$$N(B) = B \cap N(A).$$

Proof. Inclusion $B \cap N(A) \subseteq N(B)$ is trivial: if $z \in B$ nucleates all of A it certainly nucleates B. The hard part is the reverse inclusion.

Recall (III, 1.3) that the middle annihilator $\mathrm{Ann}_M(B) = \{x \in A \mid U_B x = 0\}$ is an ideal whenever B is, so $C = B \cap \mathrm{Ann}_M(B)$ is a Jordan-trivial ideal of A lying inside B ($U_C C \subseteq U_B(\mathrm{Ann}_M B) = 0$). Hence $C = 0$ by

(*) $U_B z = 0$ for $z \in B \Rightarrow z = 0$.

We can apply this together with the fact that associator maps are Jordan derivations (I.3.8) to see

(**) $[x, y, B] = 0$ for $[x, y, A] \subset B \Rightarrow [x, y, A] = 0$

since under these hypotheses any $z \in [x, y, A]$ belongs to B and is killed by B, $U_B z \subset U_B[x, y, A] = [x, y, U_B A] - U_{B, [x,y,B]} A = 0$, and therefore by (*) $z = 0$. In particular, for $z \in N(B)$ we have $[z, B, B] = 0$ and $[z, B, A] \subset B$, so by (**) $[z, B, A] = 0$; repeating, $[z, A, B] = 0$ and $[z, A, A] \subset B$ (for $z \in B$) imply $[z, A, A] = 0$ by (**). We have shown $z \in N(B) \Rightarrow z \in N(A)$: nuclearity of z in B and resistance to annihilation by U_B force nuclearity of z in A. ∎

We can establish a similar result for the center.

4.2 (Local Central Inheritance Theorem.) If an alternative algebra A is B-semiprime for an ideal B, then the center of B is

$$C(B) = B \cap C(A).$$

Proof. Once more $B \cap C(A) \subset C(B)$ is obvious, and by the previous result we at least know the central elements of B are nuclear in A, $C(B) \subset N(B) \subset N(A)$. It remains to show they commute with A. But using our standard argument we see that for any $c \in C(B)$

$$U_B [c, A] = [c, U_B A] - U_{B, [c, B]} A = 0$$

since $[c, B] = 0$ and commutator maps are Jordan derivations by I.3.7. Applying (*) we see $[c, A] = 0$ and c is central in A. ∎

For globally semiprime algebras we have

4.3 (Nuclear Inheritance Theorem) If A is a semiprime alternative algebra then the nucleus of any ideal $B \triangleleft A$ is

$$N(B) = B \cap N(A). \blacksquare$$

4.4 (Central Inheritance Theorem) If A is a semiprime alternative algebra then the center of any ideal $B \triangleleft A$ is

$$C(B) = B \cap C(A). \blacksquare$$

4.5 Corollary. Any associative ideal B in a semiprime alternative algebra A is nuclear in A. $\blacksquare$

We say A is *purely alternative* if it has no nuclear ideals; a typical example would be a Cayley algebra (or a direct sum or product of such). By 4.5, a semiprime alternative algebra is purely alternative iff it contains no associative ideals.

For the corresponding local notion, we say A is *B-purely alternative* for an ideal B if there are no nuclear ideals of A lying inside B. For these we can strengthen the Prime Nucleus Theorem to

4.6 (Semiprime Nucleus Theorem) If an alternative algebra A is B-semiprime and B-purely alternative for an ideal $B \triangleleft A$, then the nucleus and center of B coincide:

$$N(B) = C(B)$$

Proof. We must show every $n \in N(B)$ commutes with B, i.e. all commutators $[n,b]$ vanish. Both n and $[n,b]$ belong to $N(B) \subset N(A)$ by B-semiprimeness (see II.1.8 and 4.1), so by X the ideal generated by $[n,b]$ is $\hat{I}([n,b]) = \hat{A}[n,b]\hat{A}$. Since this ideal lies in B (recall $n \in B \triangleleft A$), if it is nuclear in A it must vanish by B-pure alternativity, and we will have $[n,b]=0$ as desired.

X

Now $[\hat{A}[n,A]\hat{A}, A, A] = [\hat{A}\{[n,A\hat{A}] - A[n,\hat{A}]\}, A, A]$ $\subset [\hat{A}[n,\hat{A}], A, A] = [n,A][A,A,A]$ (because $[n,\cdot]$ is a derivation and because of Nuclear Slipping II.1.6), so nuclearity of $\hat{I}([n,b])$ will follow if all $m = [n,a][x,y,z]$ vanish. The ideal $\hat{I}(m)$ lies in B since $n \in B \Rightarrow m \in B$, so by B-semiprimeness if this ideal is trivial it must vanish and $m=0$ as desired.

X

By $\hat{I}(m)$ will be trivial if m is a trivial nuclear element.

X X

But $m = [n, a[x,y,z]] - a[n,[x,y,z]] \in [N(A), A] - A[N(A), [A,A,A]]$ is nuclear by II.1.8, and it is trivial since $mwm = [n,a][x,y,z] \cdot w \cdot [n,a][x,y,z]$ $[n,a][x,y,z]\{[w,[n,a]] + [n,a]w\}[x,y,z] = \{[x,y,z][n,a][x,y,z]\} \cdot [w,[n,a]] + \{[n,a][x,y,z][n,a]\} \cdot w[x,y,z]$ (everything takes place in the associative subalgebra generated by w, $[x,y,z]$, and the nucleus), where by linearized II.1.9 $[x,y,z][n,a][x,y,z] = -[a,y,z][n,x][x,y,z] = 0$ and $[n,a][x,y,z][n,a] = -[n,a][a,y,z][n,a] = 0$.

by Nuclear Slipping II.1.6

Thus m is a trivial nuclear element as desired. ∎

The associative and purely alternative parts

It is easy to divide an algebra into a purely associative and a purely alternative part and analyze these parts separately.

The *associator ideal* $Ass(A)$ of any linear algebra A is the ideal generated by all associators $[x,y,z]$. We always have

$$Ass(A) = \hat{A}[A,A,A] = [A,A,A]\hat{A}.$$

The two expressions coincide by the Associator Identity II.2.7 ($x[y,z,w] \equiv -[x,y,z]w$ modulo $[A,A,A]$). The first of these is a left ideal since $A\{\hat{A}[A,A,A]\} \subset (A\hat{A})[A,A,A] - [A,\hat{A},[A,A,A]] \subset \hat{A}[A,A,A]$, dually the second is a right ideal, so their common value is a two-sided ideal. Clearly it is generated by the associators. Note A is associative iff $Ass(A) = 0$.

We next consider the maximal nuclear ideal of A (which, by Zorn, always exists). This can be described element-wise as the set of properly nuclear elements, where $z \in A$ is *properly nuclear* if it is nuclear and all multiples za stay nuclear,

$$[z,A,A] = [zA,A,A] = 0.$$

4.7 (Proper Nuclearity Lemma) The following are equivalent for a nuclear element $z \in N(A)$ in an alternative algebra A:

(i) all za for $a \in A$ are nuclear

(ii) all az for $a \in A$ are nuclear

(iii) $z[A,A,A] = 0$

(iv) $[A,A,A]z = 0$.

(v) $z\,Ass(A) = 0$

Proof. (i) $\Leftrightarrow$ (iii) since $[zA, A, A] = z[A,A,A]$ by Nuclear Slipping II.1.6, (iii) $\Leftrightarrow$ (v) since $zA(A) = z\{[A,A,A]\hat{A}\} = \{z[A,A,A]\}\hat{A}$. Dually (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi), and (i) $\Leftrightarrow$ (ii) since always $za - az = [z,a] \in [N, A] \subset N$ by II.1.8. ∎

The properly nuclear elements constitute the *nuclear radical*, which for reasons of euphony is called Nurd(A).

4.8 (Nuclear Radical Proposition) The nuclear radical Nurd(A) is the maximal nuclear ideal of the alternative algebra A: it is an ideal contained in the nucleus, and contains all one-sided nuclear ideals of A. A is purely alternative iff Nurd(A) = 0.

Proof. Clearly Nurd(A) is a subspace by linearity of the defining conditions in 4.6. It is closed under multiplication, since if z is properly nuclear any multiple is not only nuclear but properly nuclear: za is properly nuclear since all $(za)b = z(ab) \in zA$ are nuclear if z and zA are nuclear. It contains all nuclear right ideals B (say) since $z \in B \Rightarrow zA \subset B \subset N \Rightarrow z$ is properly nuclear. ∎

The associator ideal Ass(A) and nuclear radical Nurd(A) can be thought of as the purely alternative and associative parts of A respectively.

The nucleus radical and associator ideal are orthogonal,

(4.9)
$$Nucl(A)\cdot Assoc(A) = Assoc(A)\cdot Nucl(A) = 0$$
$$[Nucl(A) = Assoc(A)^{\perp} \cap N(A)]$$

by 4.7 (v)-(vi). In particular, since two nonzero ideals in a prime algebra cannot annihilate each other, either $Assoc(A) = 0$ or $Nucl(A) = 0$:

4.10 (Corollary) A prime alternative algebra is either associative or purely alternative. ▪

We say A is unmixed if the nuclear radical and associator ideal stay apart,
$$Nucl(A) \cap Assoc(A) = 0.$$
A typical example would be a direct sum $A = D \boxplus C$ of an associative algebra D and a Cayley algebra C, where $Nucl(A) = D$ and $Assoc(A) = C$.

4.11 (Unmixed Proposition) A semiprime alternative algebra A is unmixed, and
$$Nucl(A) = Assoc(A)^{\perp}.$$
Proof. We have $[Assoc(A)^{\perp}, A, A]$ contained in $Assoc(A)$ by construction and in $Assoc(A)^{\perp}$ since the latter is an ideal. Since $B \cap B^{\perp} = 0$ for any ideal in a semiprime algebra, we have $[Assoc(A)^{\perp}, A, A] \subset Assoc(A) \cap Assoc(A)^{\perp} = 0$ and $Assoc(A)^{\perp}$ is nuclear. In this case 4.13 reduces to $Nucl(A) = Assoc(A)^{\perp}$. Again semiprimeness implies $Assoc(A) \cap Nucl(A) = Assoc(A) \cap Assoc(A)^{\perp} = 0$, so A is unmixed. ▪

4.12 (Local Nuclear Radical Inheritance Theorem) If an alternative algebra A is B-semiprime for an ideal B, then the nuclear radical of B is

$$\mathrm{Nucl}(B) = B \cap \mathrm{Nucl}(A).$$

Proof. The inclusion $B \cap \mathrm{Nucl}(A) \subset \mathrm{Nucl}(B)$ since the left side is an ideal contained in B and nucleizing all of A. Conversely, if $z \in \mathrm{Nucl}(B)$ is properly nuclear in B we claim it is also properly nuclear in A: $zA \subset N(A)$. It suffices to show $zA \subset N(B)$ since by Local Nuclear Inheritance $N(B) \subset N(A)$. But if z is in $\mathrm{Nucl}(B)$ then z and Bz lie in $N(B) \subset N(A)$, so $[B, B, zA] = [B, Bz, A]$ (Nuclear Slipping II.1.6 since $z \in N(A)$) $= 0$ (since $Bz \subset N(A)$), and zA is nuclear in B as claimed. ∎

4.13 (Nuclear Radical Inheritance Theorem) If A is a semiprime alternative algebra then the nuclear radical of any ideal B is

$$\mathrm{Nucl}(B) = B \cap \mathrm{Nucl}(A). \quad ∎$$

Pure alternativity means vanishing of the nuclear radical. Also recall that semiprimeness is inherited (IV.0.8).

4.14 (Pure Alternativity Inheritance Theorem) If A is a semiprime purely alternative algebra, so is any ideal $B \triangleleft A$. ∎

Recall that A is <u>B-purely alternative</u> if no nuclear ideals of A lie inside B, i.e. $B \cap \mathrm{Nucl}(A) = 0$.

4.15 (Local Pure Alternativity Inheritance Theorem) If A is B-semiprime and B-purely alternative for an ideal $B \triangleleft A$, then B is purely alternative. ∎

Application to algebras with d.c.c. on left ideals

A nice example of the power of this approach is the application to algebras with d.c.c. on left ideals (generalizing our results in Chapter III on algebras with d.c.c. on all inner ideals).

4.16 (Artin–Wedderburn Theorem) An alternative algebra is semiprime with d.c.c. on left ideals iff

$$A = \mathrm{Alt}(A) \boxplus \mathrm{Nucl}(A)$$

where the purely alternative part $\mathrm{Alt}(A)$ is a finite direct sum of Cayley algebras over fields, and the associative part $\mathrm{Nucl}(A)$ is a finite direct sum of artinian associative matrix algebras over division rings.

Proof. Clearly $\mathbb{C}$ and $M_n(\Delta)$ and any finite direct sum thereof has d.c.c. on one-sided ideals and is semiprime.

For the converse, assume at first that A is semiprime with d.c.c. merely on two-sided ideals.

Let S denote the "socle" of $\mathrm{Alt}(A)$, the sum of all those minimal ideals of A which lie in $\mathrm{Alt}(A)$. Since these are precisely the irreducible $M(A)$-submodules of $\mathrm{Alt}(A)$, by module theory we know S is a direct sum of irreducibles, and by the d.c.c. on two-sided ideals this sum must be finite: $S = C_1 \boxplus \cdots \boxplus C_n$. By the Minimal Ideal Theorem III.1.10 a minimal ideal is either trivial or simple, and it can't be trivial in a semiprime algebra, so all minimal ideals B are simple. By Kleinfeld's Simple Theorem B is then Cayley or associative. If $B \subset \mathrm{Alt}(A)$ then $B \cap \mathrm{Nucl}(A) = 0$ and $\mathrm{Nucl}(B) =$

$B \cap \mathrm{Nuc}(A) = 0$ by semiprimeness and Nuclear Radical Inheritance. Such B are not associative, and therefore the socle S of $\mathrm{Assoc}(A)$ is a finite direct sum $C_1 \boxplus \ldots \boxplus C_n$ of Cayley algebras C_i. Since each C_i is unital, so is S. Then by ___ $\mathrm{Assoc}(A) = S \boxplus S'$ where $S' = S^{\perp} \cap \mathrm{Assoc}(A)$ is an ideal in A. If S' were nonzero it would contain a minimal ideal B by the d.c.c. where $B \subset \mathrm{Assoc}(A)$ would imply $B \subset S$ by definition of the socle whereas $B \subset S \cap S' = 0$ is impossible. Therefore $S' = 0$ and $\mathrm{Assoc}(A) = S \boxplus S' = S$ is unital, so by ___ again

$$A = S \boxplus S^{\perp} = \mathrm{Assoc}(A) \boxplus \mathrm{Nucl}(A).$$

since $\mathrm{Assoc}(A)^{\perp} = \mathrm{Nucl}(A)$ by 4.11.

Up to now we have used only the d.c.c. on two-sided ideals. In dealing with the associative part of A we need the full force of the d.c.c. As a direct summand, $\mathrm{Nucl}(A)$ inherits semiprimeness and the d.c.c. on one-sided ideals, therefore is a semiprime artinian associative algebra so by the associative Artin–Wedderburn Theorem is a finite direct sum of matrix algebras. $\boxtimes$

4.17 Remark. Notice how conceptually clean the proof is. We break the algebra into purely alternative and associative pieces Ann(A) and Nucl(A), describe these, and show A is their direct sum.

The only unfortunate aspect is that the method heavily depends on prior classification of all simple algebras. In some sense it is unsatisfactory to have to use simple algebras to classify semisimple algebras, and it is especially unsatisfactory to have to assume the general theory of simple algebras when one is dealing only with the (presumably) more manageable class of algebras with d.c.c. ∎

4.18 Corollary. If $\mathfrak{A}$ is semiprime with d.c.c. on left ideals then $\mathfrak{A}$ is unital and has d.c.c. and a.c.c. on both left and right ideals ☐

We can also use our results to compare radicals.

4.19 (Radical Equality Theorem) If $\mathfrak{A}$ is an alternative algebra with d.c.c. on left ideals then the radicals

$$P(\mathfrak{A}) = S(\mathfrak{A}) = L(\mathfrak{A}) = Nil(\mathfrak{A}) = Rad(\mathfrak{A}) = JK(\mathfrak{A}) = BMc(\mathfrak{A})$$

all coincide with the maximal nilpotent ideal.

Proof. Always $P \subset S \subset L \subset Nil \subset Rad \subset JK \subset BMc$ (see 0.00). Since $\mathfrak{A}/P(\mathfrak{A})$ is semiprime with d.c.c. we have seen it is a direct sum of unital simple algebras, so by definition $P(\mathfrak{A}) \supset BMc(\mathfrak{A})$. Thus all radicals coincide. By Zhevlakov-Slater Nilpotence 0.00 we know $P(\mathfrak{A})$ is the maximal nilpotent ideal. ☐

4.20 Remark. Again it is unsatisfactory to have to use our knowledge of simple and semisimple algebras to describe radicals. One would expect a simple direct treatment of radicals as in the associative case (or as in the case of alternative algebras with d.c.c. on quadratic ideals in V.0.00). The trouble as always is the refractory nature of one-sided ideals. ☐

Prime Inheritance

The method of middle annihilation used in semiprime and nuclear inheritance applies just as well to prime inheritance. Again we can get by with local primeness. We say $\mathfrak{A}$ is $\mathfrak{B}$-*prime* for an ideal $\mathfrak{B} \triangleleft \mathfrak{A}$ if no orthogonal ideals of $\mathfrak{A}$ lie inside $\mathfrak{B}$: if $\mathfrak{K}, \mathfrak{D}$ are nonzero ideals of $\mathfrak{A}$ contained in $\mathfrak{B}$ then their product $\mathfrak{K}\mathfrak{D}$ is also nonzero. Equivalently, if $\mathfrak{D}$ is a nonzero ideal of $\mathfrak{A}$ contained in $\mathfrak{B}$ then $\mathrm{Ann}_R(\mathfrak{D}) \cap \mathfrak{B} = 0$ and $\mathrm{Ann}_L(\mathfrak{D}) \cap \mathfrak{B} = 0$ (though perhaps there are parts of its annihilator lying outside $\mathfrak{B}$).

4.21 (Local Prime Inheritance Theorem) If an alternative algebra $\mathfrak{A}$ is $\mathfrak{B}$-prime for an ideal $\mathfrak{B}$, then $\mathfrak{B}$ is prime as an algebra.

Proof. If $\mathfrak{A}$ is $\mathfrak{B}$-prime it is certainly $\mathfrak{B}$-semiprime, so by Semiprime Inheritance ([illegible]) we know $\mathfrak{B}$ is at least semiprime. ¶ We begin by showing that if $\mathfrak{K} \triangleleft \mathfrak{B}$ is an ideal in $\mathfrak{B}$ then its annihilator $\mathrm{Ann}(\mathfrak{K}) \cap \mathfrak{B}$ an ideal in all of $\mathfrak{A}$ even if $\mathfrak{K}$ isn't,

$$(*) \quad \mathrm{Ann}(\mathfrak{K}) \cap \mathfrak{B} \triangleleft A.$$

To show primeness we will show that $\mathrm{Ann}_R(\mathfrak{K}) \cap \mathfrak{B} = 0$ for any nonzero ideal $\mathfrak{K} \triangleleft \mathfrak{B}$. First note that all annihilators $\mathrm{Ann}_L(\mathfrak{K}) \cap \mathfrak{B} = \mathrm{Ann}_R(\mathfrak{K}) \cap \mathfrak{B} = \mathrm{Ann}(\mathfrak{K}) \cap \mathfrak{B}$ coincide when $\mathfrak{B}$ is semiprime, since for example $\mathrm{Ann}_L(\mathfrak{K}) \cap \mathfrak{B} \subset \mathrm{Ann}_R(\mathfrak{K}) \cap \mathfrak{B}$: $\mathfrak{K} \cdot \{\mathrm{Ann}_L(\mathfrak{K}) \cap \mathfrak{B}\} \subset \mathfrak{K} \cap \mathrm{Ann}_L(\mathfrak{K})$ (since $\mathfrak{K} \triangleleft \mathfrak{B}$) must vanish since it is a trivial ideal in $\mathfrak{B}$, $\{\mathfrak{K} \cap \mathrm{Ann}_L(\mathfrak{K})\}^2 \subset \mathrm{Ann}_L(\mathfrak{K}) \cdot \mathfrak{K} = 0$. By symmetry, it suffices for $(*)$ if $\mathrm{Ann}(\mathfrak{K}) \cap \mathfrak{B}$ is a left ideal in $\mathfrak{A}$. This follows from $\mathrm{Ann}_L(\mathfrak{K}) \triangleleft_L \mathfrak{A}$, or equivalently

$$[\mathfrak{A}, \mathrm{Ann}_L(\mathcal{L}), \mathcal{L}] = 0,$$

which results (by 4.1(**)) from $\mathcal{B}$-semiprimeness plus $[\mathrm{Ann}_L(\mathcal{L}), \mathcal{L}, \mathfrak{A}] \subset \mathcal{B}$ plus $[\mathrm{Ann}_L(\mathcal{L}), \mathcal{L}, \mathcal{B}] = 0$.

If $\mathcal{D} = \mathrm{Ann}_R(\mathcal{L}) \cap \mathcal{B}$ weren't zero it would be a nonzero ideal of $\mathfrak{A}$ contained in $\mathcal{B}$ by (*), so by $\mathcal{B}$-primeness $\mathrm{Ann}_L(\mathcal{D}) \cap \mathcal{B} = 0$. But $\mathcal{L}$ is nonzero and contained in $\mathrm{Ann}_L(\mathcal{D}) \cap \mathcal{B}$ ($\mathcal{L}\mathcal{D} = 0$ if $\mathcal{D} \subset \mathrm{Ann}_R(\mathcal{L})$), so $\mathcal{D}$ must be zero and $\mathrm{Ann}_R(\mathcal{L}) \cap \mathcal{B} = 0$ for all nonzero ideals $\mathcal{L} \triangleleft \mathcal{B}$. ∎

In particular, if $\mathfrak{A}$ is entirely prime so are all ideals.

4.22 (Prime Inheritance Theorem) If $\mathfrak{A}$ is a prime alternative algebra, so is any ideal $\mathcal{L} \triangleleft \mathfrak{A}$. ∎

IX. 4.1 Problem Set on Nuclear Inheritance

We outline an alternate approach to nuclear inheritance which avoids Jordan methods.

1\. Show that if $B \triangleleft A$ and $n \in N(B)$, $a \in A$, $b, c \in B$ then $c[n,b,a] = [n, bc, a] = -[n, b, a]c$ is an alternating function of b, c. $= [c, [b, n], a]$

2\. If $z = [n, b, a]$ as above, show $z^2 - z \circ B = U_z B = 0$ so $\hat{B} z = z \hat{B}$ is a trivial ideal of B. Conclude that if B is semiprime then $[N(B), B, A] = 0$.

P One way to be assured B is semiprime is to assume A is B-semiprime (using Semiprime Inheritance V. 0.00). We can avoid Semiprime Inheritance by the following detour. For $z = [n, b, a]$

4\. as above show $z[B, B, B] = 0$, $z(U_B B) = 0$, $z(B c^2) = 0$. Show $2BB^2 \subset [B, B, B] + U_B \hat{B} + B \cdot U_B 1$ and conclude $2\{(zB)B\}B = 0$. If $B \cap Ann_L(B) = 0$ (as when A is B-semiprime)

conclude $2z = [z, B, B] = 0$, and therefore $[z, B] = [z, B, B] = 0$ and $z \in C(B)$. Use this and Exercise 1 to show $[z, B, A]B = 0$, so $B \cap Ann_L(B) = 0$ implies $[z, B, A] = 0$.

5\. Whenever $w \in B \triangleleft A$ satisfies $wB = Bw$ and $[w, B, A] = 0$ show $Bw = wB \triangleleft A$; if w is trivial in B show Bw is trivial, conclude that if A is B-semiprime then $w = 0$. Use this to prove $[N(B), B, A] = 0$ when A is B-semiprime.

3 Alternately, for $n \in N(B)$ and $a \in A$ show $R = A_{n,a}(B)$ and $K = Ker\, A_{n,a}$ are ideals in B with $KR = 0$, so $K \cap R \triangleleft B$ is trivial. If $B \cap Ann_L(B) = 0$ show $K = Ann_L(R)$. Show that $r^2, r \circ B, U_r B, U_B r, U_{r,B} B$ all lie in $K \cap R$ for $r \in R$. Deduce that if B is semiprime then $R = 0$ and $[n, a, B] = 0$.

Returning to where we took the detour, assume B is an ideal in A and $n \in B$ satisfies $[n, B, A] = 0$.

6 Show $[n, A, A]B = 0$, and conclude that $[n, A, A] = 0$ if $B \cap \mathrm{Ann}_L(B) = 0$.

Establish the Nuclear Inheritance Theorem 1.

7 If $c \in C(B)$ show $[c, A]B = 0$, so $[c, A] = 0$ if $B \cap \mathrm{Ann}_L(B) = 0$. Establish the Central Inheritance Theorem 3.

P We can similarly establish the one-sided theorems without Jordan methods. Throughout we assume B is a left ideal, $C = \mathrm{Ker}(B)$ the maximal two-sided ideal inside it.

8. As in Exercise 1, show $c[n, b, a] = [n, bc, a] = [c, [n, b], a] = -[n, b, a]c$ for $a \in A$, $b \in B$, $c \in C$, $n \in N(B)$ vanishes if $b = c \in C$.

9 For $z = [n, b, a]$ show $z \in C$ has $z^2 = z \circ C = U_z C = 0$, so $\hat{C}z = z\hat{C}$ is a trivial ideal of C. Conclude that $[N(B), B, A] = 0$ if A is B-semiprime (using Semiprime Inheritance). An alternate method avoids Semiprime Inheritance; show $C[N(B), C, A] = 0$ using $[N(B), C] \subset N(C) \subset N(A)$, so if A is B-semiprime $[N(B), C, A] = 0$; then show $C[N(B), B, A] = 0$, so $[N(B), B, A] = 0$ in the B-semiprime case (using $[N(B), B, A] \subset C$).

10. When $[N(B), C, A] = 0$ show $[N(B), A, A]C = 0$ as in Exercise 6. If $B \cap \mathrm{Ann}_L(C) = 0$, conclude $N(B) \subset N(A)$.

11. Derive the One-sided Nuclear Inheritance Theorem.

12. If A is B-semiprime for a left ideal B, show $[C(B), A]\, H(B) = 0$. Deduce $C(B) = B \cap C(A)$ when A is $H(B)$-semiprime (or even just $I([C(B), A])$-semiprime).

IX.4.2 Problem Set on Associative and Pure Alternative Parts

Although the associator ideal and nucleus radical are ideals in the original algebra which usually form a direct sum $A \supseteq \mathrm{Asso}(A) \boxplus \mathrm{Nucl}(A)$, they don't always add up to the whole algebra. Another way to break A into associative and purely alternative pieces is by means of quotients: define the associative part of A to be $A/\mathrm{Asso}(A)$, and the purely alternative part to be $A/\mathrm{Nucl}(A)$ (despite the fact that these are homomorphic images rather than subalgebras of A).

1. Show $A/\mathrm{Asso}(A)$ is the maximal associative image of A. Despite its name, $A/\mathrm{Nucl}(A)$ is not always purely alternative (give an example where A is solvable of index 2); show, however, that it is at least maximal in the sense that if A/B is purely alternative then $B \supseteq \mathrm{Nucl}(A)$.
2. If A is semiprime show $A/\mathrm{Nucl}(A)$ contains no associative ideals, in particular is semiprime. If A is unmixed show $A/\mathrm{Nucl}(A)$ is purely alternative.
3. Prove the Unmixed Theorem: An unmixed alternative algebra is a subdirect sum

$$A \subseteq A/\mathrm{Asso}(A) \boxplus A/\mathrm{Nucl}(A)$$

of its associative and purely alternative parts (which are the maximal associative and purely alternative images of A respectively).

This is the natural or intrinsic decomposition of a semiprime algebra into its associative and purely alternative parts.

of a semiprime algebra into a semidirect sum of an associative and a purely alternative algebra. (Compare with 3.9).

These notions of associative and purely alternative parts can be used to re-derive the Artin–Wedderburn Theorem 4.16. Their advantage is that a quotient A/B automatically inherits d.c.c., whereas we had to resort to subterfuge to assure ourselves that the ideals Nucl(A) and Assoc(A) inherited the d.c.c.

5. If A is semiprime with d.c.c. on left ideals, and B is a maximal ideal of A containing Assoc(A) but missing Nucl(A), show A/B is a semiprime associative algebra with d.c.c. on left ideals; show Nucl(A) is embedded as an ideal in A/B, and deduce Nucl(A) is a finite direct sum of simple artinian matrix algebras, so in particular Nucl(A) is unital.

Remark: we can choose Nucl(A)+ maximal among ideals missing Assoc(A), but it is not clear whether Assoc(A) is maximal among ideals missing Nucl(A).

6. Under the same hypotheses show the purely alternative part A/Nucl(A) is semiprime with d.c.c. Show it is a finite direct sum of Cayley algebras (using the Minimal Ideal Theorem and Kleinfeld's Simple Theorem). Deduce that Assoc(A) is a finite direct sum of Cayley algebras, in particular is unital.

7. Derive the Artin–Wedderburn Theorem by showing that a minimal ideal in a semiprime algebra A belongs either to Assoc(A) or Nucl(A), so that if Assoc(A) ⊞ Nucl(A) is unital its complement contains no minimal ideals of A.

IV.1.3 II.1.3 Problem Set on Nucleus and Center

How far the nucleus is from being central is measured by the ideal Curd(A) generated by all commutators $[a,n]$ for $a \in A$ and $n \in N(A)$: Curd(A) = 0 iff all $[a,n] = 0$ iff $N(A) = C(A)$.

1. Show $\mathrm{Curd}(A) = \hat{A}[A,N(A)] = [A,N(A)]\hat{A}$.

2. Show $\mathrm{Curd}(A) \subset \mathrm{Nurd}(A) \Longleftrightarrow \mathrm{Curd}(A)\cdot A(A) = 0 \Longleftrightarrow [A,N(A)]\cdot [A,A,A] = 0 \Longleftrightarrow [\mathrm{Curd}(A),A,A,] = 0$. Usually Curd(A) will be contained in Nurd(A). Show that although $[\mathrm{Curd}(A),A,A]$ may not always be zero, at least it is always contained in $N(A)$.

3. Show any $m = [a[x,n],b,c]$ (for $a,b,c,x \in A$, $n \in N(A)$) is a trivial element of the nucleus. Conclude that either $\mathrm{Curd}(A) \subset \mathrm{Nurd}(A)$ or else there is a trivial ideal $I(m)$ where m is contained in $\mathrm{Curd}(A) \cap A(A) \cap N(A)$. (We do not claim all of $I(m)$ is contained in $N(A)$; see SN 2 in the next problem set).

4. Deduce Slater's General Nuclear Theorem: If $\mathrm{Curd}(A) \cap A(A) \cap \mathrm{Nurd}(A)$ contains no trivial elements then $\mathrm{Curd}(A) \subset \mathrm{Nurd}(A)$, so that if A is also purely alternative then $N(A) = C(A)$.

5. Deduce Slater's Nuclear Theorem : If A is semiprime then $\mathrm{Curd}(A) \subset \mathrm{Nurd}(A)$. If A is semiprime and purely alternative then its nucleus and center coincide, $N(A) = C(A)$.

6. Deduce $N(A) = C(A)$ also if A has no associative ideals, or is semiprime with no associative images (=purely exceptional).

7. If A has no trivial nuclear elements show: (i) A is unmixed, (ii) $\mathrm{Curd}(A) \subset \mathrm{Nurd}(A)$, (iii) $N(A) = C(A)$ if A is purely alternative, (iv) $A(A) \cap N(A) = A(A) \cap C(A)$, (v) $[N(A), A(A)] = 0$.

IX.4 Problem Set on Slater's Nuclear Conjectures

Michael Slater has made the following conjectures about the distance of the nucleus from the center in the purely alternative case:

(SN 1) $\quad$ $\mathrm{Curd}(A) \cap A(A)$ is zero or contains trivial nuclear ideals

(SN 2) $\quad$ $\mathrm{Curd}(A) \subset N(A)$ or else $\mathrm{Curd}(A) \cap A(A)$ contains trivial nuclear ideals

(SN 3) $\quad$ A unmixed $\Rightarrow \mathrm{Curd}(A) \subset N(A)$

(SN 4) $\quad$ If $\mathrm{Nurd}(A)$ is commutative without nilpotent elements, then $N(A) = C(A)$.

(SN 5) $\quad$ A purely alternative $\Rightarrow N(A) = C(A)$

1. Show in general $1 \Rightarrow 2 \Rightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$ for alternative algebras. In the previous problem set we saw (SN 4) always holds.

2. Let $g(x_i, y_i, z_i; w_j; n_j) = \Pi_{i=1}^{r}[x_i, y_i, z_i]\ \Pi_{j=1}^{s}[w_j, n_j]$ $(r, s \geq 1)$ for $x_i, y_i, z_i, w_j \in A$ and $n_j \in N(A)$. Show g is independent of the order and association of the factors $[x_i, y_i, z_i]$ and $[w_j, n_j]$. Show its values lie in $\mathrm{Curd}(A) \cap N(A)$.

3. Show g is an alternating function of its arguments x_i, y_i, z_i, w_j, which vanishes if one of these variables lies in $N(A)$. If

$w_s = w_s' w_s''$ show $g(x,y,z;w_s;n) = g(x,y,z;w_s';n)w_s'' + g(x,y,z;w_s'';n)w_s'$ $+ g(x,y,z;w_s';n_s'')$ where $n_s'' = [w_s'', n_s]$. Conclude that if g vanishes on a set of generators x,y,z,w for A modulo N(A), it vanishes everywhere. ?

4. Conclude that if A is finitely generated mod N(A) then $g = 0$ for large enough r and s. Show SN 2-5 hold if A is finitely generated mod N(A). ✓

5. If A is generated mod N(A) by 3 elements, show Curd(A) $\subset$ N(A).

IX. 4.5 Problem Set on Prime Inheritance

We give an alternate proof of prime inheritance, based on the nuclear radical rather than Jordan methods.

left $\mathcal{L}$-prime for a

1. If $\mathfrak{A}$ is $\mathcal{L}$-prime for an ideal $\mathcal{L}$ (or left ideal $\mathcal{L}$ in the sense that there are no orthogonal left ideals of $\mathfrak{A}$ inside $\mathcal{L}$) show either $\mathcal{L} \cap A(\mathfrak{A}) = 0$ or $\mathcal{L} \cap Nucl(\mathfrak{A}) = 0$; conclude either $\mathcal{L}$ is properly nuclear or $\mathcal{L}$ is semiprime purely alternative without one-sided associative ideals.
2. If $\mathfrak{L}, \mathfrak{D}$ are orthogonal left ideals of $\mathfrak{A}$ contained in an ideal $\mathcal{L}$, show $\mathfrak{L}\mathcal{L}, \mathfrak{D}\mathcal{L}$ are orthogonal ideals of $\mathfrak{A}$ contained in $\mathcal{L}$ if $\mathcal{L}$ is properly nuclear; if $\mathfrak{A}$ is $\mathcal{L}$-prime, conclude $\mathfrak{L} = 0$ or $\mathfrak{D} = 0$.
3. If $\mathfrak{L}, \mathfrak{D}$ are orthogonal left ideals of $\mathfrak{A}$ contained in an ideal $\mathcal{L}$, show $Ker(\mathfrak{L}), Ker(\mathfrak{D})$ are orthogonal ideals of $\mathfrak{A}$ contained in $\mathcal{L}$; if $\mathfrak{A}$ is $\mathcal{L}$-prime conclude $Ker(\mathfrak{L}) = 0$ or $Ker(\mathfrak{D}) = 0$. If $\mathcal{L}$ is semiprime purely alternative, deduce $\mathfrak{L} = 0$ or $\mathfrak{D} = 0$.
4. Prove the Theorem. If $\mathfrak{A}$ is $\mathcal{L}$-prime for an ideal $\mathcal{L}$, then it is left and right $\mathcal{L}$-prime.
5. Deduce the One-sided Primeness Theorem.
6. Use the associator derivation formula 0.00 to show $[\mathfrak{A}, \mathfrak{L}, Ann(\mathfrak{L})]\mathcal{L} = 0$ if $\mathfrak{L} \triangleleft \mathcal{L} \triangleleft \mathfrak{A}$; if $\mathfrak{A}$ is $\mathcal{L}$-prime conclude $Ann(\mathfrak{L}) \cap \mathcal{L} = Ann_\ell(\mathfrak{L}) \cap \mathcal{L} = Ann_r(\mathfrak{L}) \cap \mathcal{L} \triangleleft \mathfrak{A}$. Deduce the Prime Inheritance Theorem.

As with semiprimeness, primeness will not in general be inherited by a mere one-sided ideal (Same example:

$\mathfrak{A} = M_n(\Phi)$ is simple associative, hence prime, but the left ideal $\mathcal{L} = \mathfrak{A}e_{11}$ is not even semiprime, much less prime).

Under certain conditions primeness will be inherited.

7. If $\mathcal{L}$ is a left ideal of $\mathfrak{A}$, where $\mathfrak{A}$ is left-$\mathcal{L}$-prime and $\mathcal{L} \not\subset \mathrm{Nucl}(\mathfrak{A})$, show $\mathcal{L}$ is prime as an algebra (Use Exercise 1 and 5?)

[$\mathcal{L}_0 = I[\mathcal{L},\mathcal{L},\mathfrak{A}] \subset \mathrm{Ker}(\mathcal{L}) \subset \mathcal{L}$, $\mathcal{L}_0 = I[\mathcal{L},\mathcal{L},\mathcal{L}]$; $\mathcal{D}_0 = I[\mathcal{D},\mathcal{D},\mathcal{D}]$ are nonzero $\mathfrak{A}$-ideals

8. If $\mathcal{L} \subset \mathrm{Nucl}(\mathfrak{A})$ is a left ideal of $\mathfrak{A}$, where $\mathfrak{A}$ is left $\mathcal{L}$-prime and $\mathcal{L} \cap \mathrm{Ann}_r(\mathcal{L}) = 0$, show $\mathcal{L}$ is prime as an algebra. (Copy Exercise 2).

In addition to primeness, weak primeness is inherited. We say $\mathfrak{A}$ is *$\mathcal{L}$-weakly prime* for an ideal $\mathcal{L} \triangleleft \mathfrak{A}$ if $\mathfrak{A}$ is $\mathcal{L}$-semiprime and $\mathcal{L}$-purely alternative, and if in addition $\mathfrak{A}$ is *$\mathcal{L}$-torsion free* (the elements β of $\mathcal{L} \cap C(\mathfrak{A}) = C(\mathcal{L})$ act injectively on $\mathfrak{A}$: $\beta a = 0 \Rightarrow \beta = 0$ or $a = 0$).

9. Show $\mathfrak{A}$ is $\mathcal{L}$-weakly prime iff $\mathcal{L}$ is weakly prime as an algebra.

$\mathfrak{A}$ itself is *weakly prime* if it is $\mathfrak{A}$-weakly prime.

IX. 4.6 Problem Set on One-sided Inheritance

We can carry many of our inheritance results over to one-sided ideals. If B is a one-sided ideal in A we say A is *B-semiprime* if there are no trivial two-sided ideals of A contained in B. Since any such ideal would lie in the maximal two-sided ideal $Ker(B)$ contained in B, B-semiprimeness is just $Ker(B)$-semiprimeness.

1. If B is a left ideal and A is B-semiprime, use 4.1 (**) to show $[N(B), B, A] = 0$. Then show further $[N(B), A, A] = 0$ by first verifying $[N(B), A, A] \subset Ker(B)$ (using left bumping).

2. Deduce the Local One-sided Nucleus Inheritance Theorem: If an alternative algebra A is B-semiprime for some one-sided ideal B, then the nucleus of B is

$$N(B) = B \cap N(A).$$

3. For the center we need $H(B)$-semiprimeness rather than just $Ker(B)$-semiprimeness. Show $[C(B), A] H(B) = 0$ for any left ideal B of A. If $H(B) \cap Ann_r(H(B)) = 0$ show $[C(B), A] = 0$.

4. Deduce the Local One-sided Central Inheritance Theorem: If an alternative algebra A is $H(B)$-semiprime for a one-sided ideal B, then the center of B is

$$C(B) = B \cap C(A),$$

5. Mimic the two-sided proof to establish the Local One-sided Nuclear Radical Inheritance Theorem: If an alternative algebra A is B-semiprime for a one-sided ideal B, then

$$Nrad(B) = B \cap Nrad(A).$$

6. Establish the Global One-sided Inheritance Theorem: If A is a semiprime alternative algebra then for all one-sided ideals B

$$N(B) = B \cap N(A)$$
$$C(B) = B \cap C(A)$$
$$\mathrm{Nucl}(B) = B \cap \mathrm{Nucl}(A).$$

7. If A is B-semiprime and B-purely alternative for a one-sided ideal B in A (in the sense that no one-sided nuclear ideal of A lies inside B, $B \cap \mathrm{Nucl}(A) = 0$) show B is purely alternative.

8. It is not in general true that a one-sided ideal in a semiprime algebra is semiprime; give a simple counterexample. However, such counterexamples exist only in the presence of associativity: in the purely alternative case semiprimeness is inherited. If A is B-semiprime show $C \cap \mathrm{Ker}(B) = 0$ for a trivial ideal $C \triangleleft B$, then show C is nuclear in B, and use this to deduce the One-sided Semiprime Inheritance Theorem: If an alternative algebra A is B-semiprime and B-purely alternative for a one-sided ideal B, then B is semiprime and purely alternative as an algebra.

9. If $BC = 0$ for left ideals B, C of A show $N(B)N(C) = 0$. Derive the One-sided Primeness Theorem: If A is a prime alternative algebra then the product BC of two nonzero left (or right) ideals B, C is again nonzero.

If $\mathfrak{L}$ is a one-sided ideal in an associative algebra $\mathfrak{A}$, there is no reason why $\mathfrak{L}$ should contain a nonzero ideal of $\mathfrak{A}$ (e.g. $\mathfrak{A} = M_n(\Phi)$ simple, $\mathfrak{L} = \mathfrak{A}e_{11}$). However, if $\mathfrak{A}$ is a Cayley algebra its only one-sided ideals are $\mathfrak{L}=0$ and $\mathfrak{L}=\mathfrak{A}$, so every nonzero one-sided ideal is a nonzero two-sided ideal. This holds more generally.

10. Show that if $\mathfrak{A}$ is $\mathfrak{L}$-semiprime for a nonzero left ideal $\mathfrak{L}$ then $\mathrm{Ker}(\mathfrak{L})=0$ implies $\mathfrak{L} \subset \mathrm{Nucl}(\mathfrak{A})$. Derive the Proposition: If an alternative algebra $\mathfrak{A}$ is $\mathfrak{L}$-semiprime and $\mathfrak{L}$-purely alternative for some nonzero one-sided ideal $\mathfrak{L}$ then $\mathfrak{L}$ contains a nonzero two-sided ideal of $\mathfrak{A}$. In particular, if $\mathfrak{A}$ is semiprime and purely alternative then every nonzero one-sided ideal contains a nonzero two-sided ideal.

Thus one-sided ideals are not far from being two-sided in the purely alternative case.

II.4.7 Problem Set on Minimal Left Ideals

We wish to obtain for one-sided ideals an analogue of the Minimal Ideal Theorem. We will make free use of nuclear inheritance.

1. If B is a minimal one-sided ideal in A, show the ideal generated by $[B,B,A]$ is either 0 or B. If $I([B,B,A]) = B$ deduce B is actually a minimal two-sided ideal of A contained in $A(A)$. In this case show B is a simple Cayley algebra, in particular is unital: $B = Ae = eA$ for a central idempotent e.
2. If $I([B,B,A]) = 0$ show B is associative; if B is not trivial show A is B-semiprime, and use Local One-sided Nuclear Inheritance (Problem Set 4.6) to conclude $B \subset Nuc(A)$.
3. If B is a non-trivial minimal left ideal of A contained in $Nuc(A)$, show $I(B) = B\hat{A} = BI(B) \subset Nuc(A)$, and that B is a minimal left ideal of the algebra $I(B)$. Conclude that $B = I(B)e = Ae$ for a nuclear idempotent e.
4. If B is as in Ex. 3, and if $Ann_L(I(B)) = 0$, show $I(B)$ is simple. Show the annihilator condition is necessary by constructing a 2-dimensional example with B a field and $I(B) = A$ not semiprime.
5. Prove the Minimal Left Ideal Theorem. A minimal left ideal B of an alternative algebra A is either

   (i) trivial, $B^2 = 0$;

   (ii) a simple two-sided ideal contained in $A(A)$ and isomorphic to a Cayley algebra, $B = I(B) \subset A(A)$;

   (iii) a minimal left ideal of a nuclear ideal $I(B)$, B minimal in $I(B) = BI(B) \subset Nuc(A)$.

   In cases (ii)–(iii) B is of the form $B = Ae$ for a nuclear idempotent e.

6. Prove Hopkins Theorem. If an alternative algebra A has d.c.c. on left ideals, then any non-nilpotent left ideal B contains a nonzero idempotent e. [Hint: $A/P(A)$ is semiprime with d.c.c., $\bar{B}$ a nonzero left ideal, hence contains a minimal left ideal $\bar{A}\bar{e}$; we can lift $\bar{e}$ in $\bar{B} \cong B/B\cap P$ through the nil ideal $P(A)$ to an idempotent $e \in B$].

7. Show directly that the Jacobson–Kleinfeld radical $JK(A)$ and Jacobson–Smiley radical $Rad(A)$ coincide for alternative algebras with d.c.c. on left ideals. [Hint: Assume $Rad(A)=0$, so by semiprimeness a minimal left ideal in $JK(A)$ gives rise to a nuclear idempotent $e \in JK(A)$.]

It has been conjectured that if A is purely alternative with d.c.c. on left ideals then A has d.c.c. on right ideals. (By 4.12 this is known if A is semiprime).

Here we said x is nonsingular if x is not a zero divisor, $xy = 0$ or $yx = 0$ implies $y = 0$, i.e. L_x and R_x are injective linear transformations.

II.4.9 Problem Set — Goldie Theory

Here we develop an analogue of the Goldie theory for associative algebras with a.c.c. An arbitrary linear algebra A is a left order in $\tilde{A}$ (or $\tilde{A}$ is a classical left quotient algebra of A) if $A \subset \tilde{A}$ and

(1) all nonsingular nuclear elements of A are invertible and nuclear in $\tilde{A}$,

(2) all elements of $\tilde{A}$ may be expressed as left quotients $x^{-1}y$ for $x, y \in A$ and x nonsingular and nuclear.

1. Show A has a left quotient algebra iff for all elements (resp. nonsingular nuclear elements) $x \in A$ and nonsingular nuclear $y \in A$ there exist $x' \in A$ (resp. nonsingular nuclear x') and nonsingular nuclear $y' \in A$ with $y'x = x'y$. [If A is alternative, show x, y, y' nuclear and nonsingular imply x' is nuclear and right nonsingular ($ax' = 0 \Rightarrow a = 0$); show it is left nonsingular ($x'a = 0 \Rightarrow a = 0$)]

2. Show a classical left quotient algebra is unique up to A-isomorphism (if it exists), i.e. if $A \subset \tilde{A}_1$ and $A \subset \tilde{A}_2$ are left quotient algebras there is an isomorphism $\tilde{A}_1 \to \tilde{A}_2$ which is the identity on A.

3. Prove the Theorem If A is a prime purely alternative algebra of characteristic $\neq 3$ then A has a left quotient algebra which is simple with d.c.c. on left ideals (in fact, is a Cayley algebra over its center with no proper left ideals at all).

Note that, unlike the associative case, we need assume no chain conditions on A at all.

4. Deduce from the previous theorem that if A is prime purely alternative, any nonzero left ideal B has zero left and right annihilators. Prove this directly by establishing the

Lemma: If B is a nonzero left ideal in A where A is B-semiprime and B-purely alternative (see 0.00 and 0.[illegible]) then B contains a nonzero ideal of A.

5. Formulate and prove a Goldie Theorem for abstract prime alternative al[illegible] #3. What can you do for semiprime purely alternative algebras?

III. # 10 #AII.7 Problem Set on Semiprime Centers

We want to show that a semiprime alternative algebra of characteristic $\neq 3$ has a center.

1. If A is semiprime show Im $3 \cap$ Ker $3 = 0$. Show $\bar{A} = {}^{A}/_{\text{Ker } 3}$ is nonzero iff $3A \neq 0$; show $\bar{A}$ is still semiprime but has no 3-torsion, and imbeds in a semiprime Ω-algebra $\tilde{A}$ for some ring of scalars with $1/3 \in \Omega$. Conclude $N(\tilde{A}) \neq 0$ and $N(\bar{A}) \neq 0$ and $N(A) \neq 0$ if $3A \neq 0$.
2. Use Problem Sets 00 and 00 to show that either $3A(A) = 0$ or
$C(\text{Assoc}(A)) = N(\text{Assoc}(A))$
$C(A(A)) = N(A(A)) \neq 0$.
3. Deduce Theorem If A is a semiprime alternative algebra then either $3A = 0$ or $N(A) \neq 0$, and either $3A \subset N(A)$ or $C(A) \neq 0$.
4. Prove Proposition If B is a one-sided ideal in A and A is B-semiprime then either $3B = 0$ or $B \cap N(A) \neq 0$, and either $3B \subset N(A)$ or $B \cap C(A) \neq 0$.
5. Deduce as corollary that if $N(A)$ is a field either $3B = 0$ or $B = A$, and that if $C(A)$ is a field then either $3B \subset N(A)$ or $B = A$.

IX.4.11 ~~#AI#?8~~ Problem Set on Weakly Prime Algebras

An algebra is said to be <u>weakly prime</u> if it is semiprime, purely alternative, and faithful as a module over its center Φ (in the sense that $\alpha x = 0$ for $\alpha \in \Phi$, $x \in A$ forces $\alpha = 0$ or $x = 0$).

1. Show that any prime algebra which is not associative is weakly prime. Show that the center Φ of a weakly prime algebra is zero or an integral domain.
2. Show that an algebra with center $\Phi \neq 0$ is weakly prime iff it is a Φ-order in a weakly prime algebra over a field. Show a central algebra over a field is weakly prime iff it is semiprime purely alternative.
3. Show that a weakly prime algebra over a field of characteristic $\neq 3$ is simple, and therefore a Cayley algebra. (Use Problem Sets ~~AI#7~~ IX.2., 000 and 000).
4. Show that if A is weakly prime and $3A \neq 0$ then A has center $\Phi \neq 0$.
5. Deduce <u>Slater's Weakly Prime Theorem</u>. A weakly prime algebra with $3A \neq 0$ is an order in a Cayley algebra over a field.
6. Show that a prime algebra is either associative or weakly prime.
7. Deduce <u>Slater's Prime Theorem</u>. A prime algebra with $3A \neq 0$ is either associative or an order in a Cayley algebra.

This method of proving Slater's Prime Theorem reduces prime algebras directly to simple algebras; the basic idea is that an ideal has nonzero nucleus = center, hence hits the nucleus = center of the original, and therefore essentially contains an invertible element.

8. Prove <u>Proposition</u> If A is semiprime with center $C(A) = \Phi$ a field of characteristic $\neq 3$, then A is associative or a Cayley algebra over Φ.

At first glance this looks like a much more general theorem since one thinks of semiprime algebras as being direct (really subdirect) sums of prime algebras. However, a direct sum decomposition of A would lead to a decomposition of its center, so the condition that the center be a field prevents there being more than one direct summand, so A looks prime.