§3 Prime algebras

At only two places in Kleinfeld's Simple Theorem did we require simplicity of A as opposed to primeness: when we reduced to the case of an algebra over an infinite field, and when we concluded that a simple algebra was strongly semiprime ~~couldn't contain locally nilpotent ideals~~. The first difficulty is easily taken care of, the second is not. We can overcome the second by fiat and obtain a structure theorem for strongly prime algebras, but the question of the structure of arbitrary prime algebras remains partly open in characteristic 3.

We say A is strongly prime if it is prime and strongly semiprime, i.e. prime and contains no trivial elements. By Kleinfeld's Strong Semiprimeness Theorem all prime algebras of characteristic $\neq 3$ are automatically strongly prime.

We need a method of creating an infinite centroid without at the same destroying primeness; unlike simplicity, primeness is not preserved by arbitrary scalar extensions (see Ex. 3.4).

3.1 (Prime Polynomial Proposition) An alternative Φ-algebra A is a Φ order in the alternative $\Phi[t]$-algebra $A[t] = A \otimes_\Phi \Phi[t]$ of polynomials in t with coefficients from A. If A has centroid Γ then A has infinite centroid containing $\Gamma[t]$. If A is prime or strongly prime, so is $A[t]$.

Proof. $A \otimes_\Phi [t] = A \otimes_\Phi (\oplus_{i=0}^{\infty} \Phi t^i) = \oplus_{i=0}^{\infty} (A \otimes_\Phi \Phi t^i)$ is canonically isomorphic to $\oplus_{i=0}^{\infty} At^i = A[t]$, so we identify the polynomial algebra $A[t]$ with the scalar extension $A_{\Phi[t]}$, which always inherits alternativity. Clearly A is an order, $A[t] = \sum t^i A = \Phi[t]A$.

It is not hard to verify that $\Gamma[t] = \Gamma \otimes_\Phi \Phi[t]$ is contained in the centroid of $A[t]$, since in general if $\Omega = \oplus_\alpha \Phi \omega_\alpha$ is free over Φ then $\Gamma(A)_\Omega$ is imbedded in $\Gamma(A_\Omega)$ via the canonical imbedding $End_\Phi(A) \otimes \Omega \to End_\Omega(A \otimes_\Phi \Omega)$ (the endomorphism corresponding to $T = \sum T_\alpha \otimes \omega_\alpha$ is $\tilde{T}(a \otimes \omega) = \sum T_\alpha(a) \otimes \omega_\alpha \omega$, thus if $\tilde{T} = 0$ then $\tilde{T}(a \otimes 1) = 0$ for all

$a \in A$ so we have $\sum T_\alpha(a) \otimes \omega_\alpha = 0$ and all $T_\alpha(a) = 0$ by Φ-freeness, hence all $T_\alpha = 0$ and $T = 0$).

If A is prime so is $A[t]$ since if $\tilde{B}\tilde{C} = 0$ for nonzero t-invariant ideals $\tilde{B}, \tilde{C}$ in $A[t]$ then $BC = 0$ where B, C are the (nonzero) ideals in A of leading coefficients of $\tilde{B}, \tilde{C}$ (if b_n, b'_m lead $\tilde{b}, \tilde{b}'$ in $\tilde{B}$ then $ab_n, b_n a, \alpha b_n + \beta b'_m$ are zero or the leading coefficients of $a\tilde{b}, \tilde{b}a, \alpha t^{m}\tilde{b} + \beta t^{n}\tilde{b}' \in \tilde{B}$, using

The polynomial algebra $A[t]$ also inherits strong semiprimeness from A: if $z = a_0 + a_1 t + \ldots + a_n t^n$ is a trivial element of $A[t]$, $z \cdot A[t] \cdot z = 0$, then its leading coefficient a_n is trivial in A, $a_n \cdot a \cdot a_n = 0$ for all $a \in A$, since if $z \cdot a \cdot z$ vanishes so must its coefficient $a_n \cdot a \cdot a_n$ of t^{2n}.

Putting primeness and strong semiprimeness together, $A[t]$ inherits strong primeness from A. ∎

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Next we wish to pass to an algebra over a field. Note that the elements of the centroid $\Gamma(A)$ are injective on a prime algebra A, $\gamma(a) = 0$ implies $\gamma = 0$ or $a = 0$. Indeed, if γ is in the centroid then $\mathrm{Im}\,\gamma$ and $\mathrm{Ker}\,\gamma$ are ideals in A which kill each other, $\mathrm{Im}\,\gamma \cdot \mathrm{Ker}\,\gamma = 0$, since $(\gamma A) \cdot (\gamma^{-1} 0) = A \cdot \gamma(\gamma^{-1} 0) = A \cdot 0 = 0$. Thus if A is prime either $\mathrm{Im}\,\gamma = 0$ (and $\gamma = 0$) or $\mathrm{Ker}\,\gamma = 0$ (and γ is injective). In particular, Γ is an integral domain, so has a field of fractions $\tilde{\Gamma}$, and A is torsion-free as Γ-module, so is imbedded in its $\tilde{\Gamma}$-module of "fractions" a/γ ($a \in A$, $0 \neq \gamma \in \Gamma$). This module of fractions is made into a $\tilde{\Gamma}$-algebra $\tilde{A}$ with the obvious multiplication $a/\gamma \cdot a'/\gamma' = aa'/\gamma\gamma'$; the resulting "algebra of fractions" is canonically isomorphic to the scalar extension $A \otimes_\Gamma \tilde{\Gamma}$ via $a \otimes \delta/\gamma \to \delta(a)/\gamma$ (this is easily checked if one notes that every element of $A \otimes_\Gamma \tilde{\Gamma}$ may be written as $a \otimes 1/\gamma$, since by finding a common denominator we can write any finite collection of fractions in the form δ_i/γ and $\sum a_i \otimes \delta_i/\gamma = \sum \delta_i(a_i) \otimes 1/\gamma = a \otimes 1/\gamma$

We call this algebra of fractions the *centroid closure* $\tilde{A}$ of A. It inherits primeness and strong primeness.

3.2 (Prime Fraction Proposition) Every prime (resp. strongly prime) alternative algebra A with centroid Γ is embedded as a Γ-order in its centroid closure $\tilde{A}$, which is a prime (resp. strongly prime) alternative algebra over the field of fractions $\tilde{\Gamma}$ of Γ. Furthermore, $C(\tilde{A})$ may be embedded in $\tilde{\Gamma}$.

Proof. Primeness of the fraction algebra comes about because if $\tilde{B}$, $\tilde{C}$ are nonzero ($\tilde{\Gamma}$-) ideals of fractions with $\tilde{B}\tilde{C} = 0$ then $B = \tilde{B} \cap A$, $C = \tilde{C} \cap A$ are nonzero ideals in A satisfying $BC = 0$ (nonzero because $\tilde{\Gamma}B = \tilde{B}$: if $b/\gamma \in \tilde{B}$ then $b = \gamma \cdot b/\gamma \in \tilde{B} \cap A = B$ and $b/\gamma = 1/\gamma \cdot b \in \tilde{\Gamma}B$).

Strong semiprimeness comes about because if $\hat{z} = z/\gamma$ is trivial in $\hat{A}$ then z is trivial in A: $\hat{z} a \hat{z} = zaz/\gamma = 0 \Rightarrow zaz = 0$ for all $a \in A$.

An element c/γ belongs to the center $C(\hat{A})$ of $\hat{A}$ iff c belongs to the center $C(A)$ of A, since $[c/\gamma, a/\alpha] = [c,a]/\gamma\alpha$ and $[c/\gamma, a/\alpha, b/\beta] = [c,a,b]/\gamma\alpha\beta$ vanish for all $a/\alpha, b/\beta \in \hat{A}$ iff $[c,a]$ and $[c,a,b]$ vanish for all $a, b \in A$. For semiprime algebras $c \to L_c$ imbeds $C(A)$ in $\Gamma(A)$ (note $L_c = 0 \Rightarrow cA = 0 \Rightarrow \hat{I}(c)^2 = 0 \Rightarrow c = 0$), hence $c/\gamma \to L_c/\gamma$ imbeds $C(\hat{A}) = C(A)/\Gamma$ in $\hat{\Gamma} = \Gamma/\Gamma$. ∎

Putting these two results together, we are allowed to pass to prime algebras over an infinite field.

3.3 Corollary. Any prime (resp. strongly prime) alternative Φ-algebra A is a Φ-order in a prime (resp. strongly prime) alternative algebra B over an infinite field $\Omega \supset C(B)$.

Proof. We first embed A as an order in $A[t]$ (with infinite centroid $\Gamma \supset \Phi[t]$) and then embed $A[t]$ as an order in its centroid closure $B = \widehat{A[t]}$ (with centroid containing the infinite field $\tilde{\Gamma}$). Here $C(B)$ can be isomorphically imbedded in $\tilde{\Gamma}$ so we may replace $\tilde{\Gamma}$ by an equivalent infinite field Ω which actually contains $C(B)$ set-theoretically, $\Omega \supset C(B)$.

Certainly A is an order in B, $\Omega \cdot A = \Omega\,\Phi[t] \cdot A = \Omega \cdot A[t] = \tilde{\Gamma} \cdot A[t] = \widehat{A[t]} = B$. Furthermore, B inherits primeness or strong primeness from A since these are inherited at both stages (polynomials and fractions) of the construction of B. ∎

Recall that there were only two places in Kleinfeld's Simple Theorem where we needed simplicity rather than primeness; when we passed to an algebra over an infinite field, and when we used strong semiprimeness to apply the Nucleus Existence Theorem.

3.4 (Strong Prime Theorem) If A is a prime alternative algebra then either

(i) A is a prime associative algebra

(ii) A is an order in a Cayley algebra

(iii) A is prime of characteristic 3 but is not strongly prime (so it has trivial elements and locally nilpotent ideals, therefore is not prime associative or a Cayley order), and satisfies $[x,y]^4 = 0$ for all $x, y \in A$.

Proof. First assume A is prime over an infinite field Ω containing the center $C(A)$. We repeat the argument of Kleinfeld's Simple Theorem. Assume A is not associative as in (i), so $N(A) = C(A) \subset \Omega$ (by the Prime Nucleus Theorem). Then $x \in A$ will be degree 2 over Ω if $[x,y]^4 \neq 0$ for some y by Hall's Identity, and A will be degree 2 over Ω (hence a Cayley algebra as in (ii)) if $[x_0, y_0]^4 \neq 0$ for some x_0, y_0 by a Zariski density argument, so we have (i) or (ii) unless $[x,y]^4 \equiv 0$. In this case A cannot be strongly prime, since a strongly semiprime algebra must satisfy one of the 3 conditions of the Nucleus Existence Theorem and our algebra fails all 3.

Now consider an arbitrary prime A. By 3.3 we can embed A as an order in a prime algebra B over an infinite field $\Omega \supset C(B)$; by the above B is either associative (whence A is associative as in (i)), or Cayley (whence A is an order in the Cayley algebra B as in (ii)), or else $[x,y]^4 = 0$ and identically on A too, and A is not strongly prime by 3.3; if it has trivial elements it must have locally nilpotent ideals since $\mathrm{Triv}(A) \subset L(A)$ by II.0.00, and it must have characteristic 3 by Kleinfeld's Strong Semiprimeness Theorem, whence (iii)). $\Box$

3.5 Remark. We can actually say more about case (iii): the set of nilpotent elements forms a locally nilpotent ideal K.
Here K is a nonzero prime algebra which is $^{(1)}$ locally nilpotent with trivial elements (hence neither associative nor Cayley), (2) characteristic 3, (3) has no nucleus whatsoever ($N(A)=0$), (4) satisfies $[x,y]^4 \equiv 0$.

such that A/K is commutative associative without nilpotent elements.

To see that the set K of nilpotent elements forms an ideal, it suffices to prove that αx is nilpotent if $\alpha \in \Phi$ and $x \in K$, $x+y$ is nilpotent if $x, y \in K$, xy and yx are nilpotent if $x \in K$, $y \in A$. Thus it suffices to consider two elements at a time and show that for any $A_0 = \Phi[x,y] \subset A$ the nilpotent elements form an ideal in A_0. But now by Artin we are dealing with an associative algebra A_0 satisfying a polynomial identity $[x,y]^4 \equiv 0$ not satisfied by any matrix algebra $M_n(\Omega)$ for $n \geq 2$ (note $[e_{11}, e_{12} - e_{21}]^4 = (e_{12}+e_{21})^4 = e_{11}+e_{22}$), so a theorem of Amitsur asserts $A_0/\mathrm{Nil}(A_0)$ is a subdirect sum of fields and hence contains no nilpotents, so the nilpotent elements of A_0 constitute the ideal $\mathrm{Nil}(A_0)$.

Clearly $\bar{A} = A/K$ has no nilpotent elements (if $\bar{x}^n = \bar{0}$ then $x^n \in K$ nil implies x nil and $x \in K$), so $[\bar{x},\bar{y}]^4 = \bar{0}$ implies $[\bar{x},\bar{y}] = 0$ and by II. 4.2 $\bar{A}$ is commutative associative without nilpotent elements.

The fact that K is locally nilpotent rather than merely nil follows from results about polynomial identities (see Appendix O.OO). K contains all the trivial elements of A, so we have (1). We have (2) and (4) as K inherits its characteristic and identities from A. The lack of nucleus (3) follows from the fact that $N(K)$ cannot contain nilpotent elements, since a semiprime algebra can't contain nilpotent central elements and $N(K) = C(K)$ by not-associative primeness.

primeness First, K inherits primeness from A by the Prime

X Inheritance Theorem 4.00.

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IP Note we cannot go on in (iii) to say A itself is nil: if K is prime but locally nilpotent, not associative or Cayley, ← no space →

then $A = \Phi 1 + K$ is still prime, not associative or Cayley, but also no longer nil — it is an extension of the locally nilpotent K by the commutative associative domain $A/K = \Phi$.

The question whether there are any algebras of type (iii) is a major unsolved problem in the theory of alternative algebras. It would be discouraging if such algebras existed. ∎

By assuming from the start that our prime algebra is strongly semiprime, we avoid having to consider prime algebras in which all $[x,y]^4$ vanish.

3.6 (Slater's Strongly Prime Theorem) A strongly prime alternative algebra is either a prime associative algebra or an order in a Cayley algebra over a field. ∎

In particular, all prime algebras of characteristic $\neq 3$ are associative or Cayley orders.

P Prime algebras are easy to construct. A set S of nonzero elements of an alternative algebra is ***m-closed*** if whenever s, t lie in S the product $\hat{I}(s)\hat{I}(t)$ of the ideals they generate contains another element u of S. For example, S can be any multiplicatively closed subset ($s, t \in S \Rightarrow u = st \in S$) or chain $S = \{x_n\}$ with $x_{n+1} \in (x_n A$ (if $n \geq m$ then $\hat{I}(x_n) \subset \hat{I}(x_m)$, and $u = x_{n+1} \in \hat{I}(x_n)^2 \subset \hat{I}(x_n)\hat{I}(x_m)$) $\boxed{\text{or more generally with } x_{n+1} \in \hat{I}(x_n)^2}$

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3.7 Lemma. If S is m-closed and P an ideal maximal with respect to exclusion of S, $P \cap S = \emptyset$, then P is a prime ideal in A and A/P is a prime algebra.

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P Proof. There exist ideals excluding S, for example the zero ideal (because $0 \notin S$!). Therefore we can invoke Zorn to obtain a maximal such ideal $P \triangleleft A$. If $\bar{B}, \bar{C}$ are nonzero ideals in $\bar{A} = A/P$ with $\bar{B}\bar{C} = \bar{0}$ their preimages B, C in A are ideals properly containing P with $BC \subset P$, but by maximality $B > P$, $C > P$ imply B, C hit S, i.e. $s \in B$, $t \in C$, so some $u \in \hat{I}(s)\hat{I}(t) \subset BC \subset P$, contrary to our assumption $S \cap P = \emptyset$. Consequently no B, C exist and $\bar{A}$ is prime. ∎

From this it is easy to see a semiprime algebra is built out of prime algebras.

3.8 (Decomposition of Semiprime into Prime) An alternative algebra is semiprime iff it is a subdirect product of prime algebras.

Proof. $A \cong \prod_i A_i$ is a subdirect product iff we have epimorphisms $A \xrightarrow{\pi_i} A_i$ with kernels K_i such that $\bigcap K_i = 0$.

If the A_i are prime (or even semiprime) then A is semiprime, since if $B^2 = 0$ for $B \triangleleft A$ we would have $\pi_i(B)^2 = 0$ for $\pi_i(B) \triangleleft A_i$ and therefore $\pi_i(B) = 0$ by semiprimeness of A_i, so $B \subset K_i$ and $B \subset \bigcap K_i = 0$.

Conversely, if A is semiprime we claim $\bigcap \{P \mid P \triangleleft A \text{ is a prime ideal}\} = 0$ so $A \cong \prod A/P$ is a subdirect product of primes. For each $x \neq 0$ in A we can recursively construct $x_i \neq 0$ with $x_1 = x$ and $x_{i+1} \in \hat{I}(x_i)^2$ since A is semiprime and $\hat{I}(x_i) \neq 0$. Then $S = \{x_i\}$ is m-closed, so by the Lemma there exists a prime ideal P with $P \cap S = \emptyset$. In particular, $x = x_1 \notin P$, so $x \notin \bigcap P$ for any $x \neq 0$. Thus $\bigcap P = 0$. ☒

Using this we can reduce most questions about semiprime algebras to ones about prime algebras. For example, since we know all prime algebras of characteristic $\neq 3$ we have

3.9 Corollary. If A is a semiprime alternative algebra with $3A = A$ then $A \cong B \oplus C$ is a subdirect sum of a semiprime associative algebra B and an algebra C which is a direct product of Cayley orders. ☒

IX. 3 Exercises

3.5 Show that the center C of a prime ring is 0 or an integral domain, and in the latter case that the central closure $\tilde{A} = A\otimes_C \tilde{C}$ is isomorphic to the algebra of "fractions" a/c and has center just the field of fractions $\tilde{C}$ of C.

3.2 What is the relation of the centroid of the centroid closure $\tilde{A} = A\otimes_\Gamma \tilde{\Gamma}$ to the original centoid Γ ? *Show the center of $\tilde{A}$ is either 0 or is naturally isomorphic to $\tilde{\Gamma}$.*

3.3 Give an example where the centroid closure $\tilde{A}$ is degree 2 over $\tilde{\Gamma}$ but A is not degree 2 over Γ .

3.4 Give an example of a prime associative algebra with center a field Φ and an extension $\Omega \supset \Phi$ such that A_Ω no longer remains prime.

~~3.5 Does an ideal $B \lhd A$ inherit primeness from A ? (It does in the associative case)~~.

3.6 If $\mathbb{Z} \subset \Phi \subset \Gamma(A)$ show A is $\mathbb{Z}$-prime (prime as $\mathbb{Z}$-algebra, ie. as a ring) iff A is Φ-prime iff A is Γ-prime.

3.7 Show that if A is an order in a Cayley algebra $\tilde{A}$ over a field Ω, so is any one-sided ideal B of A.

3.1. Show the centroid of $A[t]$ (as $\Phi[t]$-algebra) consists of all $T(T_1, T_2, \ldots)$ defined by $T(a) = \Sigma_{i=0}^{\infty} T_i(a)t^i$ (hence $T(\Sigma_{j=1}^{n} a_j t^j) = \Sigma_{k=0}^{\infty} \{\Sigma_{i+j=k} T_i(a_j)\} t^k$) where $\{T_1, T_2, \ldots\}$ is a locally finite family of elements T_i from the centroid of A (as Φ-algebra), in the sense that for any $a \in A$ only finitely many $T_i(a)$ are nonzero. Conclude $\Gamma(A)[t] \subset \Gamma(A[t]) \subset \Gamma(A)[[t]]$ under natural identification.

3.9 If A is prime and strongly semiprime, show $a(Ab) = 0$ implies $ab = 0$. Is this true in an arbitrary prime algebra?

3.10 Does $a(Ab) = 0$ imply $(aA)b = 0$ in a prime algebra? In a prime and strongly semiprime algebra?

3.11 Is it true that A is prime plus strongly semiprime iff $a(Ab) = 0$ implies a or b is 0?

P It would be useful to have a succinct
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practical elementwise criterion for primeness.

3.8 Show that if A is prime with centroid Φ then the unital hull $\hat{A} = \Phi 1 \oplus A$ need not be prime, but its quotient $\bar{A} = \hat{A}/K$ is a unital prime algebra containing A when K is the ideal consisting of all $\gamma 1 \oplus c$ for $c \in C(A)$ and $\gamma = -L_c \in \Gamma(A) = \Phi$. In particular, $\Gamma(\bar{A}) = C(\bar{A}) = \Phi 1$.

3.12 Show that a subdirect product of strongly prime algebras is strongly prime. What about the converse? If $x \neq 0$ in a strongly prime A, is there a strongly prime ideal P such that $x \notin P$? Show these hold when $3A = A$.

IX.3.1 ~~#AII.4~~ Problem Set on Algebras with Non-nil Heart

As in associative algebras, the heart M of an alternative algebra A is the intersection of all nonzero ideals of A. (In particular, a non-trivial algebra A is simple iff it is all heart, $M = A$). If $M \neq 0$, it is the unique minimal ideal.

1. Show that if the heart M of A is not trivial, $M^2 \neq 0$, then A is prime.

2. Deduce that if the heart of A is not trivial, either A is associative or $N(A) = C(A)$.

3. If B is a nonzero ideal in a prime algebra, show $[B,A,C(B)] = 0$, $[A,A,C(B)] = 0$, $[C(B),A] = 0$. Conclude that if A is prime and $B \lhd A$ then $C(B) \subset C(A)$. (We will return to this in Section 4).

4. Show that in general if $c \in C(A)$ then $cM = 0$ or $cM = M$ (M the heart); if A is prime show $cM = M$ for all $c \in C(M)$ and hence $C(M)$ is zero or a field. If A is prime and $C(M)$ is a field with unit e, show e is the unit for A, hence $M = A$. Conclude that if A is prime and its heart M has nonzero center $C(M) \neq 0$, then $C(M)$ is a field and $A = M$ is simple.

5. Assume A is prime, not associative, and $C(M) = 0$. Show fourth powers of commutators in M are zero, so the nilpotent elements of M form an ideal $Z(M) \lhd M$. Show $Z(M) \lhd A$, conclude $Z(M) = M$ or $Z(M) = 0$. If $Z(M) = M$ then M is nil; if $Z(M) = 0$ show $M = C(M) = 0$. Conclude that if A is prime but not associative, with heart M satisfying $C(M) = 0$, then M is nil.

6. Prove the <u>Theorem</u>. If A has a non-nil heart then A is either associative or a Cayley algebra.

Using some results of the next section, we can extend this from the case of a nil-free heart to the case of a nontrivial heart. Every alternative algebra contains an *associator ideal* $A(A)$ (the ideal generated by all associators) and a *nuclear radical* $Nuc(A)$ (the maximal nuclear ideal); these ideals annihilate each other, and if $Nuc(A) = 0$ in a prime algebra then there are no associator ideals in A.

7. If $A(A)$ and $Nuc(A)$ are nonzero, show the heart M of A is trivial: $M^2 = 0$. If $A(A) = 0$ show A is associative. If A contains no associative ideals, show M is trivial or a Cayley algebra; in the latter case show $M = A$. [Hint: use the Minimal Ideal Theorem and Kleinfeld's Simple Theorem] ~~[Show M = C is a direct summand; then show its complement vanishes by definition of heart.]~~

8. Prove the Theorem. If A has nontrivial heart, $M^2 \neq 0$, then A is either associative or a Cayley algebra over a field.

9. Prove the Theorem. If A is prime and contains a minimal ideal M, then M is nontrivial and the heart of M, so A is either associative or a Cayley algebra over a field.