§2 Simple alternative algebras

In ~~the~~ this section we extend the Bruck-Kleinfeld-Skornyakov Theorem to arbitrary simple algebras. The fact that we are no longer in a division algebra forces us to modify slightly the approach of the previous section.

The first order of business is to prove that in a simple not-associative algebra A the nucleus N and center C coincide. The proof of the Nucleus = Center Theorem required cancellation, so we will give a different (and more general) proof.

close up! don't leave so much room

2.1 (Prime Nucleus Theorem). If A is a prime alternative algebra either A is associative, N(A) = A , or the nucleus and center coincide, N(A) = C(A).

Proof. Assume N(A) ≠ C(A), so N doesn't commute with everything: [A ,N] ≠ 0. We will show A is associative.

¶ The ideal generated by the nonzero subspace $M = [A, N]$ is just $\hat{A}M = M\hat{A}$. Indeed, $M \subset N$ is nuclear by II.1.8, so $A(\hat{A}M) = (A\hat{A})M \subset \hat{A}M$ shows $\hat{A}M$ is a left ideal and similarly $M\hat{A}$ is a right ideal; they coincide since $[A, M] \subset [A, N] = M$ shows $AM \subset MA + M = M\hat{A}$ and dually $MA \subset \hat{A}M$.

¶ Next, this ideal is killed by any associator $w = [[x,y,z],b,c]$. To show $w \in Ann_R(\hat{A}M)$ it suffices if $Mw = 0$, since then $(\hat{A}M)w = \hat{A}(Mw) = 0$. But $mw = [a,n]w = [a,n][[x,y,z],b,c]$ $= -[[x,y,z],n][a,b,c]$ (by the linearized Nucleus-Center Identity II.1.9′), which vanishes since by II.1.8 the

$\mathbb{P}$ By primeness $(\hat{A}M) \cdot \mathrm{Ann}_R(\hat{A}M) = 0$ forces the annihilator of the nonzero ideal $\hat{A}M$ to vanish, so all $w = [[x,y,z],b,c]$ are zero and all $n = [x,y,z]$ belong to the nucleus:

$$\boxed{(*) \quad [A,A,A] \subset N.}$$

Furthermore,

$$\boxed{(**) \quad [x,A,A][x,A,A] = 0}$$

since the nuclear element $n = [x,y,z]$ has $n[x,A,A] = [nx,A,A] = [[x,y,z]x,A,A] = [[x,y,zx],A,A]$ (right bumping) $\subset [N,A,A] = 0$ by $(*)$;

$\mathbb{P}$ These relations imply all associators $n = [x,y,z]$ are trivial nuclear elements:

$$\boxed{\begin{aligned} nan &= [x,y,z]\{a[x,y,z]\} \\ &= [x,y,z]\{-2[x,y,a] + [x,ya,z] + [x,yz,a]\} \\ &= -[x,zy,z][x,y,a] + [x,y,z][x,ya,z] + [x,y,z][x,yz,a] \\ &= 0 \end{aligned}}$$

by linearized left bumping, right bumping, and $(**)$. But by V.6.3 a trivial nuclear element generates a trivial ideal, which must vanish by primeness, so all associators n vanish and A is associative. ∎

There are certain general conditions under which an algebra necessarily has a nucleus. Using the Fourth Power Theorem as a source of nuclear elements, we have

~~2.1~~ 2.2 (Nuclear Existence ~~Proposition~~ Theorem) If $A \neq 0$ is a strongly semi-prime alternative algebra then its nucleus is nonzero, $N(A) \neq 0$.

Indeed, either (i) some fourth power $[x,y]^4 \neq 0$ is a nonzero nuclear element, (ii) some $[x,y]^2 a [x,y]^2 \neq 0$ is a nonzero nilpotent nuclear element, or (iii) A is commutative associative without nilpotent elements.

Proof. By the Fourth Power Theorem all $z = [x,y]$ have $z^4 \in N(A)$. Either (i) some z^4 is nonzero, or else all $z^4 = 0$. But if all z^4 vanish, all elements $w = z^2 a z^2$ are nilpotent nuclear elements: $w^2 = z^2 a z^4 a z^2 = 0$ by Artin if $z^4 = 0$, and $w = z^2 a z^2 = [z^2, az^2]$ is nuclear since $[w, b, c] = -[[z^2, az^2], b, c] = [z^2, az^2, bc] - b[z^2, az^2, c] - [z^2, az^2, b]c$

X (by the Associator Derivation Formula (0.00)) vanishes because $[z^2, Az^2, A] = z^2[z^2, A, A]$ (by left bumping) $= 0$ (by (1.2)). Either (ii) some $z^2 a z^2$ is nonzero nilpotent nuclear or else all $z^2 a z^2 = 0$. But if all $z^2 a z^2 = 0$ then all z^2 are trivial, so by strong semiprimeness all $z^2 = 0$. Linearization of $[x,y]^2 = 0$ yields $[x,y] \circ [x',y] = 0$ for all x, x', y. We want to prove A contains no nilpotent elements in this case, Now let z be any element with $z^2 = 0$ and let a,b be arbitrary in A. We have $U_{zaz}\, b = U_{z[a,z]}\, b$ (if $z^2 = 0$) $= R_{[a,z]} U_z L_{[a,z]}\, b$ (Right Fundamental) $= \{z[a,z]\cdot bz\}[a,z]$ (Middle Moufang) $= \{z[a,z]\cdot[b,z]\}[a,z]$ (as $z[a,z]\cdot zb = zaz\cdot zb$ $= z\{a(z^2b)\} = 0$ by Left Moufang) $= z\{[a,z][b,z][a,z]\}$ (Right Moufang) $= -z\{[b,z][a,z][a,z]\} = 0$ by the vanishing of squares of commutators $[a,z]^2 = [a,z]\circ[a',z] = 0$. Thus zaz is trivial, so $zaz = 0$ by hypothesis whenever $z^2 = 0$; but then z is trivial, so $z = 0$ whenever $z^2 = 0$.

This implies A contains no nilpotent elements. In particular, $[x,y]^4 = 0$ implies all commutators $[x,y] = 0$. Thus A is commutative without nilpotent elements, hence associative (by FII. 4.1) (and we have (iii).) ∎

Space

The Kleinfeld Strong Semiprimeness Theorem says a semiprime algebra on which 3 is injective a surjective is necessarily strongly semiprime, so

~~3.2~~ 2.3 (Theorem). A nonzero semiprime algebra on which 3 is injective or surjective has a nonzero nucleus. ∎

We now are set to prove

(2.4) (Kleinfeld's Simple Theorem) A simple alternative algebra is either associative or a Cayley algebra over its center.

Proof. We may regard the simple algebra A (a central simple) as an algebra over its centroid Γ (which is a field containing the center C of A, and coinciding with it if $C \neq 0$). Since any scalar extension $A_\Omega = A \otimes_\Gamma \Omega$ remains central simple (by Strict Simplicity II.1.5), and since if A_Ω is associative or Cayley then A was to begin with, it suffices to prove the result for A_Ω. Taking an infinite Ω (in case Γ was finite), we may assume A is an algebra over an infinite field $\Gamma \supset C(A)$ ~~the centroid Γ of A is infinite~~. This will allow us to make use of the Zariski topology on A.

Assume throughout that A is not associative. To show such an A is Cayley, we begin as in the last section from Hall's identity

$$(2.5) \qquad \alpha x^2 - \beta x + \gamma = 0$$

$(\alpha = [x,y]^2,\ \beta = [x,y]\circ[x,y'],\ \gamma = [x,y']^2$ for $y' = yx)$. Since A is no longer a division algebra we do not know $z^2 \in N$ for all commutators $z = [x,y]$, so we have no guarantee α,β,γ lie in Γ.

~~CAPITAL LETTERS INSTEAD OF UNDERLINING~~

However, we claim that if $z = [x,y]$ is invertible then we do have $\alpha = z^2$, $\beta = z \circ z'$, $\gamma = z'^2$ lying in Γ for arbitrary $z' = [x,y']$. Indeed, linearizing $y \to y + \lambda y'$ in (1.2) yields STET no need to type over!

$$z[z^2,a,b] = 0$$

$$z[z\circ z',a,b] + z'[z^2,a,b] = 0$$

$$z[z'^2,a,b] + z'[z\circ z',a,b] = 0$$

for any a,b. If z is invertible we can cancel it from the first relation to get $[z^2,a,b] = 0$ for all a,b; the second relation then becomes $z[z\circ z',a,b] = 0$, and we can again cancel z to get $[z\circ z',a,b] = 0$; the third relation then reduces to $z[z'^2,a,b] = 0$, whence $[z'^2,a,b] = 0$ by cancellation. Consequently $[z^2,a,b] = [z\circ z',a,b] = [z'^2,a,b] = 0$ for all a,b and $z^2, z\circ z', z'^2$ lie in N. Since A is prime and by hypothesis not associative, the Prime Nucleus Theorem assures us $N = C$, so that α,β,γ lie in Γ (and α is invertible since $\alpha = z^2$).

So far we know x will satisfy an equation of degree 2 over Γ as soon as some $z = [x,y]$ is invertible. Furthermore, since $z^4 \in N = C$ by the Fourth Power Theorem, and $C \subset \Gamma$ is a field, z will be invertible as soon as $z^4 \neq 0$.

Thus x will satisfy a nontrivial quadratic equation if we can just find a y such that $[x,y]^4 \neq 0$. Since A is not a division algebra there is no reason to expect very many such pairs x,y. The amazing thing is that if we can find <u>one</u> such pair x_o,y_o we will be done! Recall that we are trying to prove <u>every</u> x is quadratic. Being quadratic just means $x^2,x,1$ are linearly dependent over Γ, ie. $1\wedge x\wedge x^2 = 0$ in the exterior algebra

$\Lambda(A)$ over Γ. But $F(x) = 1 \wedge x \wedge x^2$ defines a polynomial map from A into $\Lambda(A)$. Further, by our remarks above $F(x) = 0$ whenever $[x,y_o]^4 \neq 0$. But then F vanishes on the set of x for which $G(x) = [x,y_o]^4 \neq 0$; this set is Zariski-open since it is defined by a polynomial equation, and it is non-empty since $G(x_o) \neq 0$ by choice of x_o, y_o , so because we made sure we were working over an infinite field Γ we can conclude the set is Zariski-dense. If the polynomial map F vanishes on a Zariski-dense set it vanishes <u>everywhere</u>, and all x are quadratic.

At this stage we have established that if we can find a single pair x_o, y_o for which $[x_o,y_o]^4 \neq 0$ we will know all elements are quadratic. ~~How could we possibly fail? Only if $[x_o,y_o]^4 = 0$ vanished identically.~~ / But look at the three possibilities in the Nuclear Existence Theorem 2.2 (recalling that a simple algebra is strongly semiprime by D. [illegible]): case (iii) where A is commutative associative is ruled out since A is not associative, case (ii) where A contains nonzero nilpotent nuclear elements is ruled out since our A is simple with nucleus = center, and in general a semiprime algebra cannot contain nonzero nilpotent elements in its center (0.00), therefore we must have case (i) and some $[x,y]^4 \neq 0$.

By now we know each x satisfies a nontrivial equation over Γ of degree 2; since Γ is infinite we conclude A is degree 2 over Γ (see VII. 1.6). A is certainly semiprime, so as in the last section the Equivalence and Composition Algebra Theorems show it must be Cayley. ∎

2.6 Remark. An alternate proof would not seek to prove that <u>all</u> elements x are quadratic, only that there exists at least <u>one</u> x which is <u>separable</u> quadratic. For if such an element exists we can apply Albert's More General Theorem to deduce A is associative or Cayley. As before x will be quadratic, $\alpha x^2 + \beta x + \gamma = 0$ for $\alpha, \beta, \gamma \in \Gamma$, as soon as $[x,y]^4 \neq 0$. We want a separable equation, $\beta^2 - 4\alpha\gamma \neq 0$, ie. the discriminant $\delta = \{[x,y]\circ[x,yx]\}^2 - 4[x,y]^2[x,yx]^2 \neq 0$. The only way we could fail to find a separable quadratic element is that for each x,y either $[x,y]^4 = 0$ or $\delta = 0$.

At this point we have two possible paths. We can consider the identically-vanishing product

$$\varepsilon(x,y) = [x,y]^4 \{([x,y]\circ[x,yx])^2 - 4[x,y]^2[x,yx]^2\};$$

and apply results about polynomial identities (see 0.[illegible], [illegible]), or we can use the previously-established fact that $[x,y]^4$ doesn't vanish identically to conclude via the Zariski topology that δ vanishes identically (see Problem Set 2.2); in this case a fairly simple argument shows A must be a division algebra and therefore is known. ∎

Exercises IX.2

Give an alternate proof of the Prime Nucleus Theorem as follows. Let $B = \hat{A}[A,N]$ be the ideal generated by $[A,N] \neq 0$ in a prime algebra with $N \neq C$.

2.1 Show that if x commutes with N then $B[x,A,A] = 0$, so x is nuclear. Conclude $[A,A,A] \subset N$.

2.2 Show $[x,A,A][x,A,A] = 0$

2.3 If $n = [x,y,z]$. Set $S = [A,n] \neq 0$. Show $Sn = 0$ and $\mathrm{Ann}_A(S)$ is an ideal, and conclude that $S = 0$ and $n \in C$.

2.4 Conclude from semiprimeness that all $n = [x,y,z]$ are zero, so A is associative.

IX.2 / ~~#AIT3~~ Problem Set on Reasonable Elements

Those who don't believe in the Zariski topology may give a "constructive" proof. Say x, y is a reasonable pair if $[x,y]^4 \neq 0$, and say x is reasonable if there is a y such that x, y is a reasonable pair. Assume that A is an alternative algebra over a field Φ with more than 5 elements, $|\Phi| > 5$.

1. Show that if x, y_o is reasonable and y arbitrary, there are at least two values of $\lambda \in \Phi$ for which $x, y + \lambda y_o$ is reasonable.

   From now on assume x, y reasonable $\Longrightarrow [x,y]^2 \in \Phi 1$.

2. Show that if x is reasonable then all $[x,z]^2$ and $[x,z] \circ [x,w]$ lie in $\Phi 1$.
3. Show all reasonable x are quadratic.
4. Show that if x_o, y_o is reasonable then any x satisfies relations $x^2 = \alpha 1 + \beta x + \gamma x_o$ and $x^2 = \delta 1 + \varepsilon x + \nu y_o$.
5. Show in all cases x is quadratic, $x^2 \in \Phi 1 + \Phi x$.
6. Prove the Theorem. If A is an alternative algebra over a field Φ with more than 5 elements such that

   (i) there exists at least one reasonable pair x_o, y_o

   (ii) x, y reasonable implies $[x,y]^2 \in \Phi 1$

   then A is degree 2 over Φ.
7. Deduce that all simple not-associative algebras which contain reasonable pairs are degree 2, hence are Cayley algebras over their center.

IX.22 Problem Set: Alternate proof

The purpose of this set is to establish the remark in 2.6 about algebras whose elements are all inseparable of degree 2. The path we follow is not the most direct one.

1. If A is an alternative algebra over a field Φ such that every element $x \in A$ can be written $x = \alpha 1 + z$ for $\alpha \in \Phi$ and $z \in A$ nilpotent, show every element of A is either invertible or nilpotent.

2. If every element of A is invertible or nilpotent, show the set N of nilpotent elements is closed under multiplication by A: if $a \in A$ and $z \in N$ then $az, za \in N$. Show N is closed under sums: if $z, w \in N$ then $z + w \in N$. Show N is an ideal in A, $N \triangleleft A$, and A/N is a division algebra.

3. If $z^2 = 0$ for all $z \in N \triangleleft A$ are trivial on N: $zNz = 0$. If A is strongly semiprime show any ideal $N \triangleleft A$ is too: if A has no trivial elements then $zNz = 0$ for $z \in N$ implies $z = 0$. Conclude that if A is strongly semiprime it has no ideals N nil of index 2.

4. If A is strongly semiprime and every elements is invertible or nil of index 2, ie. every x^2 is invertible or zero, show A is a division algebra.

5. If A is a unital algebra over a field Φ and $x \in A$ satisfies $x^2 + \beta x + \gamma 1 = 0$ for $\beta, \gamma \in \Phi$, show x is invertible if $\gamma \neq 0$ and nilpotent of index ≤ 2 if $\beta = \gamma = 0$. If $p(\lambda) = \lambda^2 + \beta\lambda + \gamma$ is an inseparable polynomial over Φ, show x is invertible or nilpotent of index ≤ 2.

6. Deduce that any strongly semiprime algebra, all of whose elements satisfy inseparable polynomials of degree 2 over Φ.

IX.23 Problem Set on Primitive Algebras

An associative algebra is primitive if it has a faithful irreducible representation ϕ on M, in which case $M \cong A/B$ under ψ: $a \to am$ for $B = \mathrm{Ker}\,\psi$ a maximal modular left ideal which contains no nonzero two-sided ideal.

Although the module definition does not go over, the ideal definition can be carried over to alternative algebras: an alternative algebra A is (left) primitive if it contains a left primitivity ideal (a maximal modular left ideal B with kernel $K(B) = L(A,B) = 0$; see IV.2).

1. If $L(A,B) = 0$ show (i) $B = 0$ if $BA \subset B$ and (ii) $A = B = 0$ if $B\hat{A} = A, B^2 = 0$ ($B \triangleleft_\ell A$).
2. Show that for any prime (resp. semiprime) nonassociative algebra A with $A^2 \neq 0$, the center $C(A)$ and centroid $\Gamma(A)$ contain no zero divisors (resp. nilpotent elements).
3. Show $[x,y,A] = 0$ in an alternative algebra implies $[x,y] \in N(A)$ and $[x,y]x \in N(A)$. If $[x,y]$, $[x,y]x \in C(A)$ show $[x,y]^2 = 0$. Conclude that if A is an alternative algebra with $N(A) = C(A)$ then $[x,y,A] = 0$ implies $[x,y]$ is a nilpotent element of the center; if A is semiprime then $[x,y] = 0$.
4. Assume $N(A) = C(A)$ and $L(A,B) = 0$ for $B \triangleleft_\ell A$, A semiprime. Show B is commutative ($[B,B] = 0$), then $[B^2,A,A] = 0$, then B^2 is central ($[B^2,A] = 0$), then $B = 0$. Conclude that if A is semiprime with $N(A) = C(A)$ and B is a left ideal with $L(A,B) = 0$ then B is trivial, $B^2 = 0$.

If we do not use Slinko's Theorem to conclude a simple algebra is strongly semiprime, we must consider two kinds of simple algebras A: those which are strongly semiprime, $S(A)=0$, and those with $S(A)=A$ and hence $A=N(A)$ is nil. (If we use Kleinfeld's Strong Semiprimeness Theorem we [illegible] the case [illegible] appear [illegible] characteristic 3.) We can put ~~these~~ (the above) results together to show that a general simple algebra (even nil of characteristic 3) must look something like a Cayley algebra.

5. Establish the <u>Theorem</u>. If A is simple but not associative it contains no proper one-sided ideals.

Now we return to primitivity.

6. If B is a modular left ideal and C a supplementary left ideal ($B + C = A$) then $aC \subset B \Rightarrow a \in B$. If B is maximal modular, C a (two-sided) ideal not contained in B, show $D\,C = 0$ for an ideal D implies $D \subset B$. Conclude that a primitive algebra is prime.

7. Show that if B is a left primitivity ideal for A then either $B = 0$ or $B\hat{A} = A$; conclude that $B = 0$ if $B^2 = 0$, in which case A has no proper left ideals and has a right unit; deduce A has a unit. Conclude that if B is a left primitivity ideal for A with $B^2 = 0$ then A is simple with unit.

Since we know all simple algebras with unit (they cannot be nil even in characteristic 3), show

8. <u>Theorem</u> A primitive algebra is either associative or a Cayley algebra.

Note that this strengthens Slater's Prime Theorem when A is primitive rather than merely prime.

IV.24 #AII.6 Problem Set on the Jacobson-Kleinfeld Radical

In the associative case the Jacobson radical can be defined either in terms of quasi-invertibility or in terms of primitivity. In the ~~associative~~ alternative case these two approaches lead respectively to the Jacobson-Smiley radical and the Jacobson-Kleinfeld radical; it is not clear they coincide.

Define an ideal K to be primitive in A if A/K is primitive as an algebra.

1. Show K is primitive iff $K = L(\hat{A},B)$ for some maximal modular left ideal B.
2. Conclude the intersection $\bigcap B$ of all maximal modular left ideals B contains the intersection $\bigcap K$ of all primitive ideals K .
3. If $K = L(\hat{A},B)$ is primitive show $\bar{B}$ is a maximal modular left ideal in $\bar{A} = A/K$. If $\bar{A}$ is Cayley show $B = K$ and hence $K \supset \bigcap B$. If $\bar{A}$ is primitive associative show $\bigcap \bar{B} = \bar{0}$; conclude $K \supset \bigcap B$. Conclude that $\bigcap K$ contains $\bigcap B$.
4. Theorem The intersection of all maximal modular left ideals coincides with the intersection of all (left) primitive ideals of A.

This intersection is called the Jacobson-Kleinfeld radical JK(A) of A. A is semiprimitive or JK-semisimple if JK(A) = 0.

5. Establish the Theorem. An alternative algebra A is semiprimitive iff it is a subdirect sum of primitive associative algebras and Cayley algebras over fields.

6. Show that if $A(1 - y)$ generates a proper left ideal $C \lessdot A$ then C is contained in a maximal modular left ideal B_y which excludes y. Conclude that if $x \in \bigcap B$ then all elements $y \in I(x)$ have the property that $A(1 - y)$ generates all of A as left ideal. (If y is quasi-invertible then $A(1 - y) = A$ already).

7. If $x \notin B$ for some maximal modular B with modulus e show $y + b = e$ for some $y \in I(x)$, $b \in B$; show $A(1 - y) \subseteq B$; conclude that if $x \notin B$ some $y \in I(x)$ has the property that $A(1 - y)$ generates a proper left ideal.

8. Establish the <u>Theorem</u> $x \in \bigcap B$ iff $A(1 - y)$ generates A as left ideal for all $y \in I(x)$. In particular, $\bigcap B$ contains no nuclear idempotent $e \neq 0$.

   Note that 6-8 go through in any nonassociative algebra.

9. <u>Theorem.</u> The Jacobson-Kleinfeld radical of an alternative algebra contains the Jacobson-Smiley radical,

$$JK(A) \supseteq Rad(A).$$

10. <u>Theorem</u> If A has dcc. on ~~quadratic~~ inner ideals then the Jacobson-Kleinfeld and Jacobson-Smiley radicals coincide,

$$JK(A) = Rad(A).$$

Zhevlakov
[illegible]
[illegible]

I.25 Problem Set on the Hereditary Nature of the Jacobson-Kleinfeld Radical

~~1. Show~~ Recall that A is <u>semiprimitive</u> (or JK-<u>semisimple</u>) iff it is a subdirect sum of Cayley algebras over fields and primitive associative algebras, ~~Conclude~~ and shows $JK(A) \supset \mathrm{Rad}\ A$.

~~2.~~ 1. Show that if A is semiprimitive, so is any ideal $B \lhd A$. Conclude $JK(B) \subset B \cap JK(A)$.

~~3.~~ 2. Use the Minimal Ideal Theorem ~~[illegible]~~ IV.4.3 to show that if A has d.c.c. on ideals then $JK(A) = \mathrm{Rad}\ A$, and therefore $JK(B) = B \cap JK(A)$ for $B \lhd A$ in this case.

~~4.~~ 3. Recall that A is JK-radical iff it has no proper modular left ideals. Show that if $A \xrightarrow{F} \tilde{A}$ (is an epimorphism and) ~~where~~ $\tilde{B}$ is a proper modular left ideal in $\tilde{A}$, then $F^{-1}(\tilde{B}) = B$ is a proper modular left ideal in A. Conclude that if A is JK-radical, so is any homomorphic image. If B is a proper modular left ideal in A and $A \xrightarrow{F} \tilde{A}$ an epimorphism with $B \supset \mathrm{Ker}\ F$, show $F(B) = \tilde{B}$ is a proper modular left ideal in $\tilde{A}$.

~~5.~~ 4. If $B \lhd A$ and C is a left B-ideal, show $C_a = C + aC$ is a left B-ideal with $P^+(C_a) \subset C$. If C is a maximal modular left B-ideal, show it is a left A-ideal. Show each maximal modular $C \lhd_\ell B$ can be extended to a maximal modular $\tilde{C} \lhd_\ell A$ with $\tilde{C} \cap B = C$. Conclude that if A is JK radical, so is any ideal $B \lhd A$, and $B \cap JK(A) \subset JK(B)$.

~~6.~~ 5. Prove the <u>Theorem</u> The Jacobson-Kleinfeld radical is hereditary: for all ideals $B \lhd A$ we have

$$JK(B) = B \cap JK(A) .$$