Chapter IX

Algebras Without Finiteness Conditions

1 Alternative division algebras

This chapter is devoted to the study of alternative algebras without any finiteness restrictions. We begin in this section by completely determining all alternative division algebras; they turn out to be either associative or Cayley algebras, just as before. Of course, the associative division algebras are not completely classified, but from the standpoint of alternative algebras we consider our task finished if we have reduced our problem to one about associative algebras.

We recall the Nucleus = Center Theorem II.1.10: if A is an alternative division algebra, then either A is associative or its nucleus and center coincide, N(A) = C(A). It might happen that the nucleus and center reduce to zero; this is made unlikely by the striking result of Kleinfeld that the 4^{th} power of any commutator falls in the nucleus, providing a rich supply of nuclear elements.

The key to this result is a general identity involving 2^{nd} powers of commutators.

1.1 (Second Power Lemma) Any commutator $z = [x,y]$ in an alternative algebra satisfies

(1.2) $$z[z^2,a,b] - [z^2,a,b]z = 0$$ (Second Power Identity).

Proof. We begin by proving

$$(*) \qquad [x,y,z\circ a] = z\circ[x,y,a] = 0.$$

By I.3.7-8 both $D_x = [x,.]$ and $A_{x,y} = [x,y,.]$ are derivations of Jordan products. Here $A_{x,y}$ kills $z = [x,y]$ by Artin's Theorem, which establishes the first equality. Also

$$\begin{aligned}
&[x,y,[x,y]\circ a] = [x,y,[x,y\circ a] - y\circ[x,z]] \\
&= \{-[x,y\circ a,[x,y]]\} - \{[x,y,y]\circ[x,a] + y\circ[x,y,[x,a]]\} \\
&= -\{[x,y,[x,y]]\circ a + y\circ[x,a,[x,y]]\} - \{0 - y\circ[x,a,[x,y]]\} \\
&= 0
\end{aligned}$$

since D_x, $A_{x,y}$, $A_{x,[x,y]}$ are Jordan derivations and since linearization of $[x,y,[x,y]] = 0$ yields $[x,y,[x,c]] = -[x,c,[x,y]]$.

Then $[z^2,a,[x,y,b]] = z\circ[z,a,[x,y,b]] =$ $[z,a,z\circ[x,y,b]] - [z,a,z]\circ[x,y,b] = 0$ (using $(*)$)

$$(**) \qquad [z^2, a, [x,y,b]] = 0$$

Linearizing $[[x,y]^2, x,y] = 0$ yields $[[x,y]^2,x,a] = -[[x,y]\circ[x,a], x,y]$ which vanishes by $(*)$, so $[z^2, x,a] = 0$, and dually $[z^2,y,a] = 0$. Thus by Left Bumping

$$(***) \qquad [z^2,a,xa] = a[z^2,a,x] = 0$$

$$(****) \qquad [z^2,a,(xy)a] = [z^2,a,xy] = -a[z^2,y,xa] = 0.$$

(using linearized $(***)$).

We can use the linearizations of these relations to flip factors from one term to another inside an associator:

$$\begin{aligned}
[z^2,a,zb] &= [z^2,a,(xy)b] - [z^2,a,(yx)b] \\
&= [z^2,a,x(yb)] + [z^2,b,(yx)a] \qquad \text{(by } (**), (****)\text{)}
\end{aligned}$$

$$= -[z^2,(yb),xa] + [z^2,b,y(xa)] \quad \text{(by (***),(**))}$$

$$= +[z^2,xa,yb] - [z^2,(xa),yb] \quad \text{(by (***))}$$

$$= 0.$$

From this $0 = [z^2,a,zb] = z\circ[z,a,zb] = z\circ[z,a,b]z = (z\circ[z,a,b])z = [z^2,a,b]z$ by middle and left bumping. Dually $z[z^2,a,b] = 0$, and (1.2) is established. ∎

From this Second Power Identity we obtain the Fourth Power Theorem by bumping.

1.3 (Fourth Power Theorem) In an alternative algebra the fourth power of any commutator lies in the nucleus,

$$[x,y]^4 \in N(A).$$

When $[x,y]$ is not a zero divisor, already the second power lies in the nucleus, $[x,y]^2 \in N(A)$.

Proof. The relation (1.2) $z[z^2,a,b] = [z^2,a,b]z = 0$ implies $[z^2,a,b] = 0$ for all $a,b \in A$ when $z = [x,y]$ is not a zero divisor, so $z^2 \in N(A)$ is nuclear in that case. In general, by Middle Bumping it implies $[z^4,a,b] = z^2[z^2,a,b] + [z^2,a,b]z^2 = 0$ for all $a,b \in A$ and $z^4 \in N(A)$. ∎

Thus supplied with nuclear elements, we can establish the structure of an arbitrary alternative division algebra.

1.4 (Bruck-Kleinfeld-Skornyakov Theorem) An alternative division algebra is either associative or a Cayley algebra over its center.

Proof. Assume throughout that the alternative division algebra A is <u>not</u> associative. We wish to show A is a degree 2 algebra over its center. But for any element x we can actually write down a quadratic equation it satisfies:

$$\alpha x^2 - \beta x + \gamma 1 = 0 \quad \text{(Hall's identity)} \tag{1.5}$$

$$\text{where}\begin{cases} \alpha = [x,y]^2 \\ \beta = [x,y]^2 x - [x,y]x[x,y] = [x,y]\circ[x,y'] \\ \gamma = [x,y]x[x,y]x = [x,y']^2 \end{cases}$$

(here $y' = yx$ has $[x,y'] = xyx - yxx = [x,y]x$). By Artin's theorem the above equation holds identically in x and y. We must show the coefficients lie in the center and can be chosen nontrivial.

Since a division algebra has no zero divisiors, $\alpha = [x,y]^2$ and $\gamma = [x,y']^2$ lie in the nucleus N by the Fourth Power Theorem, and so does the linearization $\beta = [x,y]\circ[x,y']$. By the Nucleus = Center Theorem and the assumed nonassociativity of A,α,β,γ lie in $N = C$.

Thus every element x satisfies a quadratic equation over C. The only trouble is that it might be a trivial equation. It will be nontrivial if $[x,y] \neq 0$ (since then $\alpha = [x,y]^2 \neq 0$), so by proper choice of y we can get a nontrivial equation unless $[x,y] = 0$ for all possible y. But such an x already ring is a finite (commutative, associative) field.

Exercises IX.1

Give an alternate proof of the 4th Power Theorem as follows.

1.1 Show (*) by writing $z \circ a = xy \circ a$

$- (ay)x - y(xa) = (L_{xy} - L_y L_x) a + (R_{xy} - R_x R_y) a$ and $A_{x,y} = L_{xy} - L_x L_y = - R_{xy} + R_y R_x$, then using left and right Moufang to prove $A_{x,y}(z \circ a) = 0$.

1.2 Deduce $[z^2,x,a] = [z^2,y,a] = 0$, and linearize to obtain $x[z^2,a,b] = [z^2,a,bx]$, $y[z^2,y,a] = [z^2,ay,b]$.

1.3 Use the HIDING TRICK to show $x[y[z^2,a,b]] = y[x[z^2,a,b]]$.

1.4 Use $(xy)[z^2,a,b] = [x,y[z^2,a,b]] + x[y[z^2,a,b]]$ to show $z[z^2,a,b] = 0$.

Still another proof was given in Problem Set III.2.1

IX.1.1 Problem Set on Domains

Go back through the proof of the Bruck-Kleinfeld-Skornyakov Theorem, making whatever additional arguments are necessary, to establish.

(Theorem) If A is an alternative algebra without zero divisors, then A is either associative or an order in a Cayley algebra.

Suggested steps are:

1. Show that if A is not associative it is of degree 2 over C = N, and semiprime.
2. Show C is an integral domain with quotient field $\tilde{C}$, and that A can be imbedded as a C-order in its central closure $\tilde{A} = A \otimes_C \tilde{C}$. Show $\tilde{A}$ has no zero divisors, and is of degree 2 over $\tilde{C}$ iff is degree 2 over C.
3. If $\tilde{A}$ is not associative but is a domain of degree 2 over a field $\tilde{C}$, show $\tilde{A}$ is a Cayley algebra, so our original A is an order in a Cayley algebra.

IX.1.2 Problem Set on the Fourth Power Theorem

1. Prove that $[x,m][y,z,w][x,n] = [x,m][x,n][y,z,w] = 0$ for all $x,y,z,w \in A$ and $n,m \in N(A)$.

2. Prove any associator $a=[x,y,z]$ satisfies $a^2 = [x,y,az] - c(az) = -B_{x,y}(az)$ for $c = [x,y]$, $B_{x,y} = L_{xy} - L_y L_x$.

3. Prove that $[[x,y]^2,z,w]^2 = 0$ holds in all alternative algebras.

4. Conclude that if A has no nilpotent elements, the square $[x,y]^2$ of any commutator lies in the nucleus.

This imporves on the Fourth Power Theorem (instead of no zero divisors we need only no nilpotents). However, we cannot generalize the Nucleus = Center Theorem from the case of no zero divisors to the case of no nilpotents: if D is a central associative division algebra and C a Cayley division algebra then $A = D \boxplus C$ has no nilpotents but has nucleus $N(A) = D \oplus \Phi$ different from A or $C(A) = \Phi \oplus \Phi$.

§2 Simple alternative algebras

In this section we extend the Bruck-Kleinfeld-Skornyakov Theorem to arbitrary simple algebras. The fact that we are no longer in a division algebra forces us to modify slightly the approach of the previous section.

The first order of business is to prove that in a simple not-associative algebra A the nucleus N and center C coincide. The proof of the Nucleus = Center Theorem required cancellation, so we will give a different (and more general) proof.

2.1 (Prime Nucleus Theorem). If A is a prime alternative algebra either A is associative, $N(A) = A$, or the nucleus and center coincide, $N(A) = C(A)$.

Proof. Assume $N(A) \neq C(A)$, so N doesn't commute with everything: $[A,N] \neq 0$. We will show A is associative.

The ideal generated by the nonzero subspace $M = [A,N]$ is just $\hat{A}M = M\hat{A}$. Indeed, $M \subset N$ is nuclear by II.1.8, so $A(\hat{A}M) = (A\hat{A})M \subset AM$ shows $\hat{A}M$ is a left ideal and similarly $M\hat{A}$ is a right ideal; they coincide since $[A,M] \subset [A,N] = M$ shows $AM \subset MA + M = M\hat{A}$ and dually $MA \subset \hat{A}M$.

Next, this ideal is killed by any associator $w = [[x,y,z],b,c]$. To show $w \in Ann_R(\hat{A}M)$ it suffices if $Mw = 0$, since then $(\hat{A}M)w = \hat{A}(Mw) = 0$. But $mw = [a,n]w = [a,n][x,y,z],b,c] = -[[x,y,z],n][a,b,c]$ (by the linearized Nucleus-Center Identity II.19'), which vanishes since by II.1.8 the nuclear element n commutes with associators $[x,y,z]$.

By primeness $(\hat{A}M)\cdot Ann_R(\hat{A}M) = 0$ forces the annihilator of the nonzero ideal $\hat{A}M$ to vanish, so all $w = [[x,y,z],b,c]$ are zero and all $n = [x,y,z]$ belong to the nucleus:

$$(\ *\ ) \quad [A,A,A] \subset N.$$

Furthermore,

$$(\ *\ ) \ [x,A,A][x,A,A] = 0$$

since the nuclear element $n = [x,y,z]$ has $n[x,A,A] = [nx,A,A] = [[x,y,z]x,A,A] - [[x,y,zx],A,A]$ (right bumping) $\subset [N,A,A] = 0$ by (*).

These relations imply all assoicators $n = [x,y,z]$ are trivial nuclear elements:

$$\begin{aligned} nan &= [x,y,z]\ \{a[x,y,z]\} \\ &= [x,y,z]\ \{-z[x,y,a] + [x,ya,z] + [x,yz,a]\} \\ &= -\ [x,zy,z][x,y,a] + [x,y,z][x,ya,z] + [x,y,z][x,yz,a] \\ &= 0 \end{aligned}$$

by linearized left bumping, right bumping, and (**). But by V.00 a trivial nuclear element generates a trivial ideal, which must vanish by primeness, so all associators n vanish and A is associative. ∎

There are certain general conditions under which an algebra necessarily has a nucleus. Using the Fourth Power Theorem as a source of nuclear elements, we have

2.2 (Nuclear Existence Theorem) If $A \neq 0$ is a strongly semiprime alternative algebra then its nucleus is nonzero, $N(A) \neq 0$. Indeed, either (i) some fourth power $[x,y]^4 \neq 0$ is a nonzero nuclear element, (ii) some $[x,y]^2 a[x,y]^2 \neq 0$ is a nonzero nilpotent nuclear element, or (iii) $A = N(A)$ is commutative associative without nilpotent elements.

Proof. By the Fourth Power Theorem all commutators $z = [x,y]$ have $z^4 \in N(A)$, so we have case (i) unless all $z^4 = 0$. If all z^4 vanish then the elements z^2az^2 $(a \in A)$ are nilpotent, since by Artin $(z^2az^2)^2 = z^2az^4az^2 = 0$; moreover, they are all nuclear since $[z^2,az^2,A] = z^2[z^2,a,A] = 0$ by left Bumping and the Second Power Identity (2.1), and any time $A_{u,v} = 0$ we have $[u,v] \in N(A)$ by the Associator Derivation Formula $[[u,v],b,c] = -A_{u,v}(bc) + A_{u,v}(b) \cdot c + b \cdot A_{u,v}(c) = 0$, so in our case $[z^2,az^2] = z^2az^2 - az^4 = z^2az^2$ are all nuclear. Thus we have case (ii) unless all $z^2az^2 = 0$. But if all z^2az^2 vanish then z^2 is trivial, so by STRONG SEMIPRIMENESS all $z^2 = 0$ and $[x,y]^2 = 0$ for all x,y. We want to show A contains no nilpotents in this case.

Quite generally, for any element z in an alternative algebra