§2. The Second Structure Theorem

The classification of simple algebras divides into two cases, according as the number of idempotents is ≤ 2 or ≥ 3. From our work on interconnectivity we expect everything with 3 or more idempotents to be associative, while the 2-idempotent case gives rise to Cayley algebras. Thus simple alternative algebras are either associative or Cayley algebras.

We say A has <u>capacity</u> n if it has unit $1 = \sum_{i=1}^{n} e_i$ a sum of n orthogonal division idempotents. (Offhand it is not clear this is an invariant of the algebra, but we will see later in the section that if $1 = \sum_{j=1}^{m} f_j$ is another decomposition into division idempotents then indeed $m = n$.) We say A has <u>connected capacity n</u> if the e_i can be chosen to be interconnected. The Theorem of the Unit 1.5 guarantees that a semisimple Artinian algebra has a capacity, and the Simple Interconnectivity Theorem VII.5.5 guarantees that any capacity of a simple algebra is connected.

Algebras of capacity 1 are just division algebras: if $1 = e_1$ is a division idempotent then by definition $A = e_1 A e_1$ is a division algebra.

By the Interconnectivity Theorem VII.5.6 any algebra of connected capacity $n \geq 3$ is associative. In particular, any simple algebra of capacity $n \geq 3$ is associative.

All that remains is capacity 2. Here we don't require simplicity or the d.c.c.

2.1 (Capacity 2 Theorem) A strongly semiprime alternative algebra of capacity 2 is either

(i) a direct sum of two division algebras

(ii) an algebra $M_2(\Delta)$ of all 2×2 matrices with entries in an associative division algebra Δ

(iii) a split Cayley algebra $\mathbb{C}(\Omega)$ over a field Ω.

Proof. Capacity 2 means A has unit $1 = e_1 + e_2$ where A_{11}, A_{22} are division algebras. If $A_{12} = A_{21} = 0$ we have $A = A_{11} \boxplus A_{22}$ as in (i).

From now on assume $A_{12} + A_{21} \neq 0$. By the Peirce Triviality Condition VII.3.22, if $x_{ij} \in A_{ij}$ is not trivial then $x_{ij}A_{ji} \neq 0$, $A_{ji}x_{ij} \neq 0$. Since there are no trivial elements in a strongly semiprime algebra, if $A_{ij} \neq 0$ contains $x_{ij} \neq 0$ then $x_{ij}z_{ji} = z_{ii} \neq 0$ and $w_{ji}x_{ij} = w_{jj} \neq 0$ for some $z_{ji}, w_{ji} \in A_{ji}$. But as soon as z_{ii}, w_{jj} are nonzero they are invertible in the division algebras A_{ii}, A_{jj}, so e_i and e_j are strongly connected according to the Strong Connection Lemma VII.5.1:

$$e_{12}e_{21} = e_{11}, \qquad e_{21}e_{12} = e_{22} .$$

If $A_{ij}A_{ij}^2 = 0$ or $A_{ij}^2A_{ij} = 0$ then the elements of A_{ij}^2 are trivial, therefore $A_{ij}^2 = 0$ by strong semiprimeness. But if $A_{12}^2 = A_{21}^2 = 0$ then A is associative by the Corollary VII.3.17 to the Peirce Associativity Criterion, since from connectivity we already have interconnectivity $A_{ij}A_{ji} = A_{ii}$. Once A is associative the e_1, e_{12}, e_{21}, e_2 form a family of matrix units, so by the Wedderburn Coordinatization Theorem (Part 1) $A \cong M_2(\Delta)$ for $\Delta \cong A_{11}$ an associative division algebra. This is case (ii).

So assume $A_{12}^2 + A_{12}^2 \neq 0$, say $A_{12}^2 \neq 0$. Fix $y_{12}z_{12} \neq 0$. By strong semiprimeness $y_{12}z_{12}$ is not trivial, we can repeat the argument we used for x_{ij} above to get u_{12}, v_{12} with $u_{12}(y_{12}z_{12}) = u_{11} \neq 0$, $(y_{12}z_{12})v_{12} = v_{22} \neq 0$, so e_1 and e_2 are Cayley-connected. By the Strong Cayley-Connection Lemma VII.6.2, e_1, e_2 are strongly Cayley-connected, giving rise to a family of Cayley matrix units by the Cayley Units Construction VII.6.3, and once we have Cayley matrix units we have a split Cayley algebra $\mathbb{C}(\Omega)$ by the Zorn Coordinatization Theorem VII.6.10 for $\Omega \cong A_{11}$ commutative. Since A_{11} is also a division ring, Ω is a field. This is case (iii). ∎

Thus we can describe simple algebras of all capacities, therefore all simple algebras with d.c.c.

2.2 (Second Artin-Zorn Structure Theorem) An alternative algebra A is simple Artinian iff it is one of

(I) an alternative division algebra D

(II) a simple Artinian associative matrix algebra $M_n(\Delta)$ for Δ an associative division algèbra, $n \geq 2$

(III) a split Cayley algebra $\mathbb{C}(\Omega)$ over a field Ω.

Proof. If A is simple Artinian of capacity 1 it is a division algebra as in (I), if it has capacity 2 it is either $M_2(\Delta)$ or $\mathbb{C}(\Omega)$ as in (II) or (III) by the Capacity 2 Theorem, and it if has capacity $n \geq 3$ it is simple associative Artinian (d.c.c. on quadratic ideals implies d.c.c. on left ideals), so is $M_n(\Delta)$ as in (II) by the Artin-Wedderburn Theorem. Alternately, it has n strongly connected idempotents $e_1, \ldots, e_n$ and therefore is a matrix algebra $M_n(\Delta)$ by the Wedderburn Coordinatization Theorem (Part 1), where $\Delta \cong e_1 A e_1$ is a division algebra if the e_i are division idempotents.

Conversely, if A is as in (I) it is trivially simple Artinian since it has no proper quadratic ideals at all. In (III) we know A is simple (by II.3.15 no proper one-sided ideals) and Artinian (by II.6.1, since quadratic ideals are Ω-subspaces and $\mathbb{C}(\Omega)$ has dimension 8 over Ω). In (II) we know $M_n(\Delta)$ is simple and Artinian in the associative sense, i.e. has d.c.c. on <u>left</u> ideals; by the Associative Example I.6.9 it has d.c.c. on <u>quadratic</u> ideals. ∎

Notice that nothing further is said about alternative division algebras. Over an algebraically closed field this case disappears, and we have

2.3 (Structure Theorem over an algebraically closed field) If A is a finite-dimensional semisimple alternative algebra over an algebraically closed field Φ, then A is a direct sum of simple ideals which are split matrix algebras $M_n(\Phi)$ or split Cayley algebras $\mathbb{C}(\Phi)$.

Proof. The division algebra D and Δ and the center Ω of $\mathbb{C}$ are finite-dimensional division algebras over an algebraically closed field Φ, which means they are Φ itself. Thus in Case I $A = \Phi 1 \cong M_1(\Phi)$, in Case II $A \cong M_n(\Phi)$, and in Case III $A \cong \mathbb{C}(\Phi)$. ∎

By a standard technique of passing to a splitting field we can classify all finite-dimensional simple algebras.

2.4 (Structure Theorem for Finite-Dimensional Algebras) A simple alternative algebra which is finite-dimensional over its center is either associative or a Cayley algebra.

Proof. We only need investigate the case of a finite-dimensional division algebra D. Any extension $D_\Omega = \Omega \otimes_\Phi D$ of a central-simple algebra D remains central simple by the Strict Simplicity Theorem III.1.5 (though not in general a division algebra if D is). Indeed, since D is algebraic we can choose Ω so D_Ω is no longer a division algebra. (If $d \notin \Phi 1$ take $\Omega = \Phi[d]$; if D_Ω were a division algebra, $\Omega[d] = \Omega \otimes \Phi[d] = \Omega \otimes \Omega$ would be a subfield, whereas such a tensor product is never a field for it has a proper homomorphism $\Omega \otimes_\Phi \Omega \to \Omega$ by $\alpha \otimes \beta \to \alpha\beta$.) D_Ω is still finite dimensional over Ω, so has d.c.c., and therefore is either of Type II or III. But if D_Ω is associative, so is D, and if D_Ω is Cayley, so is D. (D is degree 2 over Φ if D_Ω is degree 2 over Ω since if $1,x,x^2$ in D are dependent over Ω they must be dependent over Φ; being semiprime but not associative, D can only be a Cayley algebra by the Composition Algebra and Hurwitz Theorems II.2.14 and II.4.1). ∎

2.5 Example. The previous example can be used to give yet another proof of Wedderburn's Theorem on finite division algebras: a finite alternative division algebra is a (commutative, associative) field. Indeed, since finite implies finite-dimensional, we just saw D is associative or Cayley. But there are no Cayley division algebras over a finite field (any quadratic norm form $n(x)$ in 8 variables must represent zero by the Artin-Chevalley Theorem) so D is associative, and we apply the associative Wedderburn Theorem. ∎

2.6 (Second Uniqueness Theorem) No two algebras of different Types (I)-

(III) are isomorphic. Two algebras $M_n(\Delta)$, $M_{\tilde n}(\tilde\Delta)$ of Type (II) are isomorphic iff $n = \tilde n$ and $\Delta \cong \tilde\Delta$. Two algebras $\mathbb{C}(\Omega)$, $\mathbb{C}(\tilde\Omega)$ of Type (III), are isomorphic iff $\Omega \cong \tilde\Omega$.

Proof. Since the algebras of Types (II) and (III) have proper idempotents, the algebras of type (I) are precisely the division algebras. The algebras of Type (II) are precisely the simple non-division algebras which are associative, Type (III) the simple non-division algebras which are not associative. Thus algebras of different types can't be isomorphic.

From the associative Artin-Wedderburn theory we know $M_n(\Delta) \cong M_{\tilde n}(\tilde\Delta)$ iff $n = \tilde n$, $\Delta \cong \tilde\Delta$. If $\Omega \cong \tilde\Omega$ certainly the split Cayley matrix algebras $\mathbb{C}(\Omega)$, $\mathbb{C}(\tilde\Omega)$ are isomorphic. Conversely, if $\mathbb{C}(\Omega) \cong \mathbb{C}(\tilde\Omega)$ then their centers are isomorphic, $\Omega \cong \tilde\Omega$. ∎

2.7 Corollary. The capacity of a simple Artinian alternative algebra is an invariant.

Proof. A has capacity 1 iff it is a division algebra, capacity 2 not-associative iff it is not-associative, capacity $n \geq 2$ associative iff it is isomorphic to $M_n(\Delta)$ (where we know n is an invariant). ∎

This structure theory suggests (and the general structure theory in Appendix II confirms) the Metatheorem that there are really only two kinds of alternative algebras, the associative algebras and the Cayley algebras. This is another way in which alternative algebras are little removed from associative algebras.

The Capacity 2 Theorem has consequences for algebras without any chain conditions.

2.8 (Albert's Theorem) If A is a simple alternative algebra which contains an idempotent $e \neq 0,1$ then A is either associative or a split Cayley algebra.

Proof. By the Alternative Corollary VII.4.5 either $A_{10}^2 = A_{01}^2 = 0$ and A is associative, or else $A_{10}^2 = A_{01}$ and $A_{01}^2 = A_{10}$. We can't have $A_{00} = 0$ or else A_{10}, A_{01} would consist of trivial elements, hence vanish by strong semiprimeness, whereas $A = A_{11}$ would contradict $e \neq 1$. Thus A_{00} is nonzero, and simple by Simple Inheritance VII.4.8. We cannot have $A_{00}A_{01} = 0$, or else $A_{00}^2 = A_{00}(A_{01}A_{10}) = (A_{00}A_{01})A_{10} = 0$ by interconnectivity. Therefore the Peirce specialization of A_{00} on $A_{01} = A_{10}^2$ is nonzero. By simplicity it must be a monomorphism; since it kills associators and commutators by VII.3.13 and 3.11, A_{00} must be associative and commutative, therefore a field.

In particular, A_{00} has a unit e_0, so A has unit $1 = e_1 + e_0$ where e_1, e_0 are division idempotents. Once A is unital it has capacity 2 and we can apply our previous work. Since A is simple and not associative, cases (i) and (ii) of the Capacity 2 Theorem are ruled out, so we must have case (iii): $A = \mathbb{C}(\Omega)$. ∎

A more general version, which encompasses non-split as well as split algebras, is

2.9 (Albert's More General Theorem) If A is a simple alternative algebra which contains an element which is algebraic over the centroid but not purely inseparable, then A is either associative or a Cayley algebra.

Proof. If x is algebraic over the centroid Φ but not purely inseparable, it has at least 2 distinct characteristic roots (in particular, $x \neq \alpha 1$ does not lie in Φ). If $\Omega \supset \Phi$ is a splitting field for the minumum polynomial $\mu_x(\lambda)$ of x, $\mu_x(\lambda) = \prod_{i=1}^{n} (\lambda - \omega_i)^{k_i}$ for $\omega_i \in \Omega$ distinct and $n \geq 2$, then in the centroid-simple algebra $A_\Omega = \Omega \otimes_\Phi A$ the element x splits, $x = \sum_{i=1}^{n} \omega_i e_i + z_i$ for e_i nonzero supplementary idempotents and $z_i^{k_i} = 0$. Since $n \geq 2$ we see $e_1 \neq 1,0$. Therefore by the split version of Albert's Theorem, A_Ω is associative or Cayley. But if A is associative so is its subalgebra A, and if A_Ω is Cayley over then A is Cayley over Φ, as we've seen before. ∎

Note that this implies the split version: if $e \neq 1,0$ is idempotent it is algebraic with separable minimum polynomial $\mu_e(\lambda) = \lambda(\lambda-1)$.

Recall that an algebra A is <u>strictly semisimple</u> if it and all its extensions A_Ω are semisimple. We say a finite-dimensional algebra A over a field Φ is <u>separable</u> if it is a finite direct sum $A = A_1 \boxplus \cdots \boxplus A_n$ of simple algebras A_i whose centers Γ_i are separable field extensions of Φ .

2.10 (Separability Theorem) A finite-dimensional algebra over a field is strictly semisimple iff it is separable.

Proof. A is semisimple iff it is a direct sum $A = A_1 \boxplus \cdots \boxplus A_n$ of simple algebras; denote the center of A_i by Γ_i . We must show all extensions A_Ω remain semisimple iff all Γ_i/Φ are separable.

Suppose first A is strictly semisimple; since each A_Ω is semisimple its center $C(A_\Omega) = C(A)_\Omega = (\Gamma_1 \boxplus \cdots \boxplus \Gamma_n)_\Omega$ $= (\Gamma_1 \otimes_\Phi \Omega) \boxplus \ldots \boxplus (\Gamma_n \otimes_\Phi \Omega)$ must be a direct sum of fields, ie. semisimple. But if Γ_i/Φ is not separable, then with $\Omega = \Gamma_i$ we see by Separability 0.000 $\Gamma_i \otimes \Omega = \Gamma_i \otimes \Gamma_i$ contains nilpotent elements and is not semisimple. To preserve semisimplicity all Γ_i/Φ must be separable.

Now suppose A is separable; to show $A_\Omega = A_{1\Omega} \boxplus \ldots \boxplus A_{n\Omega}$ remains semisimple it suffices to show $A_{i\Omega}$ are semisimple. If $A_{i\Omega}$ had a (necessarily solvable) radical so would any extension $(A_{i\Omega})_\Sigma = (A_i \otimes_\Phi \Omega) \otimes_\Omega \Sigma = A_i \otimes_\Phi \Sigma = A_{i\Sigma}$. It therefore enough to check $A_{i\Sigma}$ remains semisimple when Σ is (say) algebraically closed. But in this case by the Separable Decomposition Theorem

$A_{i\Sigma} = \boxplus A_{ij}$ for $A_{ij} \cong A_i \otimes_{\Gamma_i} \Sigma$; since A_i/Γ_i is central simple, we know by Strict Simplicity III.1.5 that all extensions $A_{ij} \cong A_i \otimes_{\Gamma_i} \Sigma$ remain simple, so $A_{i\Sigma}$ is a direct sum of simple algebras and therefore semisimple. ∎

Exercises

2.1 Prove the following version of the Capacity 2 Lemma. If A has capacity 2, $1 = e_1 + e_2$ for division idempotents e_i, then the set of trivial elements is the annihilator ideal $Z = Z_{12} + Z_{21}$ where $Z_{ij} = \{z_{ij} \in A_{ij} \mid z_{ij}A_{ji} = 0\} = \{z_{ij} \in A_{ij} \mid A_{ji}z_{ij} = 0\}$. An element $x_{ij} \in A_{ij}$ either belongs to Z_{ij} or else there is $x_{ji} \in A_{ji}$ with $x_{ij}x_{ji} = e_i$, $x_{ji}x_{ij} = e_j$, so x_{12} and x_{21} strongly connect e_1 and e_2. If A is semiprime then $Z = 0$ and A is either a direct sum of 2 division algebras or e_1, e_2 are strongly connected.

2.2 Show that a connected capacity 2 algebra is necessarily simple.

2.3 Both Structure Theorems can be proved in one fell swoop.

Theorem A strongly semiprime algebra with capacity is a direct sum of simple associative algebras $M_n(\Delta)$ $(n \geq 2)$, split Cayley algebras $\mathbb{C}$, and division algebras D.

2.4 Show bAb for $b = \varepsilon e_{11} + e_{21}$, $A = M_n(\Delta)$ is not a left vector space over Δ unless ε is in the center of Δ.

2.5 If $B = bAb$ for $b = \varepsilon e_{11} + \varepsilon\eta e_{12} + e_{21}$, $A = M_n(\Delta)$, show B is neither a right nor a left vector space over Δ if ε, η are not in the center of Δ.

2.6 Prove that in $A = M_n(\Delta)$ each quadratic ideal contained in $e_{ii}A$ is a left vector space over Δ.

2.7 Prove that $A = M_n(\Delta)$ has d.c.c. on quadratic ideals by proving it has d.c.c. on quadratic ideals contained in $e_{11}A + \dots + e_{kk}A$ by induction on k.