4. Classification of composition algebras

A composition algebra is a *-simple alternative algebra of one of the following types: a field, a quadratic extension, a quaternion algebra, or a Cayley algebra. It can be built up from any of its nonsingular subalgebras by the Cayley-Dickson construction. Isomorphism of composition algebras reduces to equivalence of their norm form; in particular, all isotopes of a composition algebra are isomorphic. Composition algebras are either division algebras or they are split, and any two split composition algebras of the same dimension are isomorphic. We furthermore establish that if a scalar extension is a composition algebra, the original algebra is a composition algebra of the same type.

Recall that a composition algebra is a unital algebra carrying a nondegenerate quadratic norm form $n(x)$ permitting composition. Recall also the distinction between nonsingular and nondegenerate.

We have seen in the Composition Proposition 3.13 that the Cayley-Dickson process builds up composition algebras. Our next result shows all composition algebras can be obtained by this process

4.1 (Hurwitz Theorem) A composition algebra over a field Φ is either

(0) a purely inseparable extension field Ω of Φ of exponent 1 and characteristic 2, with identity involution

(I) the base field Φ, with identity involution

(II) a quadratic extension $\mathbb{C} = \Phi u + \Phi\bar{u}$ $(u + \bar{u} = 1)$

(III) a quaternion algebra $\mathcal{Q}$ with standard involution

(IV) a Cayley algebra $\mathcal{C}$ with standard involution.

Proof. The radical R of the bilinear form $n(x,y)$ is an ideal (by the Radical Proposition 2.3). By nondegeneracy of the quadratic form, $n(z) \neq 0$ for $z \in R$; such z have inverses $z^{-1} = n(z)^{-1}\bar{z}$ by the Inverse Criterion 1.17, so either $R = 0$ or R contains invertible elements and thus is all of A.

We first get rid of the case $R = A$, where n is totally singular (the polarized norm form $n(x,y)$ is identically zero). In this case $x \to n(x)$ is a ring homomorphism of A into Φ, since $n(xy) = n(x)n(y)$ and $n(x+y) = n(x) + n(x,y) + n(y) = n(x) + n(y)$, and it is a monomorphism since $n(z) = 0$ implies $z = 0$ by nondegeneracy. Thus A is a field Ω containing $\Phi 1$. Since $t(x) = n(x,1)$ vanishes identically we have $2 = t(1) = 0$, so Ω has characteristic 2, hence $x + \bar{x} = t(x) = 0$ implies $\bar{x} = x$ and the involution is the identity. Then $x^2 = x\bar{x} = n(x)1 \in \Phi 1$ so Ω is purely inseparable of exponent 1. This is case (0).

From now on we assume $R = 0$, so $n(x,y)$ is nonsingular. We will show that if B is any proper finite-dimensional nonsingular subalgebra (containing 1) there is $\ell \in B^{\perp}$ with $\ell^2 = \mu 1$ such that $\mathbb{C}(B,\mu) \subset A$. Indeed, we have $A = B \oplus B^{\perp}$ since B is finite-dimensional and nonsingular. Since B is assumed proper, $B^{\perp} \neq 0$, and since n remains nonsingular on $B^{\perp}$ we can find $\ell \in B^{\perp}$ with $n(\ell) = -\mu \neq 0$. By the *-- Necessity Proposition 3.9 we have $\mathbb{C}(B,\mu) = B \oplus B\ell \subset A$.

We now apply the Cayley-Dickson process with everything taking place inside the algebra A given at the start. We begin with $B_1 = \Phi 1$. If $B_1 = A$ we have Case I. If $B_1 \neq A$ but B_1 is nonsingular we have $B_2 = \mathbb{C}(B_1,\mu_1) = \Phi 1 \oplus \Phi i \subseteq A$ by the foregoing. If B_1 is singular (i.e. the characteristic is 2) there is $u \in A$ with $t(u) = n(u,1) = 1$ by nonsingularity of $n(x,y)$, so $B_2' = \Phi 1 \oplus \Phi u$ is a subalgebra ($u^2 = t(u)u - n(u)1 \in B_2$) and is nonsingular (if $n(z,1) = n(z,u) = 0$ for $z = \alpha 1 + \beta u$ then $\beta = \alpha = 0$ as $n(1,1) = n(u,u) = 0$ and $n(1,u) = 1$). If $B_2 = A$ we have case II. If $B_2 \neq A$ we have $B_3 = \mathbb{C}(B_2,\mu_2) \subset A$. B_3 is associative but not commutative because the involution on B_2 is nontrivial. Either $B_3 = A$, and we have case III, or $B_3 \neq A$ and we have $B_4 = \mathbb{C}(B_3,\mu_3) \subset A$. B_4 is alternative but not associative. We must have $B_4 = A$ or else $A \supset \mathbb{C}(B_4,\mu_4)$, whereas $\mathbb{C}(B_4,\mu_4)$ is not alternative since B_4 is not associative. ∎

It is important to note that we can start the Cayley-Dickson process with ANY unital subalgebra which is nonsingular relative to the norm bilinear form, applying the doubling process until we eventually reach the whole algebra.

4.2 Corollary. A composition algebra A of Type I-IV can be built up from any of its nonsingular unital subalgebras B by the Cayley-Dickson process. In particular, for any nonsingular unital subalgebra of dimension $\dim B = \frac{1}{2} \dim A$ the algebra A has the form $A = \mathbb{C}(B,\mu) = B \oplus B\ell$. Here ℓ may be chosen arbitrarily in $B^{\perp}$ with $n(\ell) = \mu \neq 0$.

Thus a Cayley algebra can be built out of ANY of its quaternion subalgebras by the Cayley-Dickson process.

4.3 Remark. By the transfinite application of the unmodified Cayley-Dickson construction in characteristic 2 we could also include the composition algebras of Type 0 under the Cayley-Dickson process: any purely inseparable extension field of exponent 1 has the form $\Omega = \Phi[\omega_1, \omega_2, \ldots]$ for some (perhaps uncountable) set of elements ω_i with $\omega_i^2 = \alpha_i \in \Phi$, and the unmodified construction takes any Γ and forms $\mathbb{C}(\Gamma,\alpha)$ $= \Gamma 1 + \Gamma \omega = \Gamma[\omega]$ with $\omega^2 = \alpha \in \Phi$. In this sense we are justified in saying that THE COMPOSITION ALGEBRAS ARE PRECISELY THOSE ALTERNATIVE ALGEBRAS BUILT UP FROM Φ BY THE CAYLEY-DICKSON PROCESS. ∎

The Type of a composition algebra is an invariant, and may be described intrinsically as follows:

Type 0 = commutative associative of dimension > 1 with trivial involution

Type I = commutative associative of dimension 1 with trivial involution

Type II = commutative associative with nontrivial involution

Type III = noncommutative associative

Type IV

We will call the algebras of Type I-IV the **ordinary composition algebras**. A purely inseparable extension of Type 0 will be called an **extraordinary composition algebra**; note that it occurs only in characteristic 2, and is the only composition algebra which can be infinite-dimensional over Φ. The ordinary composition algebras have as their *-centers just Φ, but the extraordinary algebra has *-center Ω. Furthermore, Type 0 becomes Type I when considered over its *-center Ω rather than Φ; in some sense it is unusual only becuase we considered it over the "wrong" field.

It is impossible to overstress the fact that AN ORDINARY COMPOSITION ALGEBRA HAS DIMENSION 1,2,4, or 8. An ordinary composition algebra is <u>forced</u> to be finite-dimensional. An extraordinary composition algebra (purely inseparable extension field of exponent 1) may have infinite dimension.

It is often convenient to know that if a scalar extension A_Ω is Cayley over Ω, then the original algebra A was Cayley over Φ.

4.5 (Extension Theorem) If the scalar extension A_Ω is a composition algebra of Type 0-IV over Ω, then A is a composition algebra of the same type over Φ.

Proof. We begin by showing A is a degree 2 algebra over Φ. Each x in A satisfies an equation $x^2 - t(x)x + n(x)1 = 0$, but a priori the coefficients $t(x)$, $n(x)$ only lie in Ω. On the other hand, x must satisfy <u>some</u> equation $x^2 - \alpha x + \beta 1 = 0$ over Φ since if $1, x, x^2$ in A were independent over Φ they would remain independent over Ω in $A_\Omega = \Omega \otimes_\Phi A$. Since such an equation of degree 2 is unique for $x \notin \Omega 1$ we see the trace $t(x) = \alpha$ and norm $n(x) = \beta$ actually lie in Φ, while for $x = \omega 1 \in \Omega 1 \cap A = \Phi 1$ trivially $t(x) = 2\omega$, $n(x) = \omega^2$ lie in Φ. Thus t and n restrict to linear and quadratic forms on A with values in Φ, and A is degree 2 over Φ.

Next we show A is a composition algebra. If the norm form on A were degenerate, $n(z) = n(z,A) = 0$ for some $z \neq 0$, then it would remain degenerate on A_Ω, $n(z) = n(z,A_\Omega) = 0$ (by bilinearity of $n(x,y)$), whereas by definition the norm of the composition algebra A_Ω is nondegenerate.

Finally, we show A has the same type as A_Ω. If A_Ω is commutative, associative, or has identity involution the same is true of its subalgebra A, and conversely if A has one of these properties so does its scalar extension A_Ω. Moreover, A_Ω has the same dimension $\dim_\Omega A_\Omega = \dim_\Phi A$ over Ω that A does over Φ. Therefore by (4.4) A has the same type as A_Ω. ∎

4.5 Remark. The converse is not quite true. If A has Type II-IV then $n(x,y)$ is nondegenerate on A, hence remains nondegenerate on A_Ω, so A_Ω is again a composition algebra (necessarily of the same Type). Trivially if $A = \Phi 1$ has Type I then so does $A_\Omega = \Omega 1$. But if $A = \Omega$ has Type 0 its scalar extensions need not remain composition algebras (for example, $A_\Omega = \Omega \otimes_\Phi \Omega$ has trivial elements $\omega \otimes 1 - 1 \otimes \omega$ for $\omega \in \Omega \setminus \Phi$). Once more the trouble is that such an A is really of Type I over Ω, so we should be tensoring over Ω instead of over Φ. ∎

Isomorphisms

The proof of the Hurwitz theorem makes it clear that the norm form and the unit determine the structure of the composition algebra. This leads to a basic criterion for isomorphism.

4.7 (Isomorphism Theorem for Composition Algebras) Two composition algebras over a field Φ are isomorphic iff their norm forms are equivalent.

Proof. Suppose $F: A \to \tilde{A}$ is an isomorphism of composition algebras. We claim $\tilde{n}(Fx) = n(x)$, so F is automatically an equivalence of quadratic forms. (This is really a special case of the result that the generic norm and minimum polynomial of an algebra are invariant under isomorphisums.) We have $x^2 - t(x)x + n(x)1 = 0$ in A, so applying F yields $\tilde{x}^2 - t(x)\tilde{x} + n(x)\tilde{1} = 0$ for all $\tilde{x} = F(x)$ in $\tilde{A}$. But we already know $\tilde{x}^2 - \tilde{t}(\tilde{x})\tilde{x} + \tilde{n}(\tilde{x})\tilde{1} = 0$ in $\tilde{A}$, so $[\tilde{t}(\tilde{x}) - t(x)]\,\tilde{x} = [\tilde{n}(\tilde{x}) - n(x)]\tilde{1}$ for all $\tilde{x}$. If $\tilde{x} \notin \Phi\tilde{1}$, independence gives $\tilde{t}(\tilde{x}) - t(x) = \tilde{n}(\tilde{x}) - n(x) = 0$, while if $\tilde{x} = F(x) = \alpha\tilde{1} \in \Phi\tilde{1}$ then (as F is bijective) $x = \alpha 1 \in \Phi 1$ trivially has $\tilde{t}(\tilde{x}) = t(x) = 2\alpha$, $\tilde{n}(\tilde{x}) = n(x) = \alpha^2$. Thus in all cases $\tilde{n}(Fx) = n(x)$, and F is an equivalence of norm forms.

Now suppose the quadratic forms n, $\tilde{n}$ of A, $\tilde{A}$ are equivalent under a linear bijection $F: A \to \tilde{A}$, $\tilde{n}(F(x)) = n(x)$. First consider the totally singular case of an inseparable field extension $A = \Omega$, where $n(x,y) \equiv 0$. Then $\tilde{n}(\tilde{x},\tilde{y}) = 0$ too and $\tilde{A} = \tilde{\Omega}$ is also an inseparable field. In this case F must already be an algebra isomorphism: it preserves products, $F(xy) = F(x)F(y)$, because $\tilde{n}(F(xy)) = n(xy) = n(x)n(y) = \tilde{n}(Fx)\tilde{n}(Fy) = \tilde{n}(Fx \cdot Fy)$, and

$\tilde{n}$: $\tilde{A} \to \Phi$ is injective ($\tilde{n}(\tilde{z}) = \tilde{n}(\tilde{w})$ forces $\tilde{z} = \tilde{w}$ in Ω).

From now on we assume $n(x,y)$ is nonsingular. In general F itself will not be an algebra isomorphism, but we will use a Cayley-Dickson process to build an isomorphism on larger and larger nonsingular subalgebras till we have an isomorphism on all of A. We start off with a rather pedestrian isomorphism F_1: $B_1 = \Phi 1 \to \tilde{B}_1 = \Phi\tilde{1}$. In characteristic $\neq 2$ this foothold is enough, but in characteristic 2 the subalgebra B_1 is totally singular and (as when building composition algebras) won't do for our induction step. We choose $u \in A$ with $n(u,1) = 1$ by nonsingularity, $n(u) = -\mu_1$. We can always <u>assume $F1 = \tilde{1}$:</u> if necessary replace F by its translate $F_o(x) = F(vx)$ where $F(v) = \tilde{1}$; then $n(v) = \tilde{n}(Fv) = \tilde{n}(\tilde{1}) = 1$ implies $\tilde{n}(F_o(x)) = \tilde{n}(F(vx)) = n(vx)$ $= n(v)n(x) = n(x)$ so F is again an isometry but now $F_o(1)$ $= F(v) = \tilde{1}$. Once $F(1) = \tilde{1}$ then $\tilde{u} = Fu$ also has $\tilde{n}(\tilde{u},\tilde{1})$ $= \tilde{n}(Fu,F1) = n(u,1) = 1$, $\tilde{n}(\tilde{u}) = \tilde{n}(Fu) = n(u) = \mu_1$, and $F_2(\alpha 1+\beta u)$ $= \alpha\tilde{1} + \beta\tilde{u}$ defines an isomorphism F_2: $B_2 = \Phi 1 + \Phi u \to \tilde{B}_2 = \Phi\tilde{1} + \Phi\tilde{u}$ of (now nonsingular) subalgebras. Indeed, all that is necessary for F_2 to be an isomorphism is that $F_2(u^2) = F_2(u)^2$, which follows from $u^2 = t(u)u - n(u)1 = u + \mu_1 1$ and $\tilde{u}^2 = \tilde{t}(\tilde{u})\tilde{u} - \tilde{n}(\tilde{u})\tilde{1} = \tilde{u} + \mu_1\tilde{1}$.

No matter what the characteristic, once we have an isomorphism F_i: $B_i \to \tilde{B}_i$ of proper <u>nonsingular</u> subalgebras we can enlarge F_i. Indeed, since $B_i^{\perp}$ is also nonsingular (by nonsingularity of $n(x,y)$) we can find $\ell \in B_i^{\perp}$ with $n(\ell_i) = -\mu_i \neq 0$ and build $B_{i+1} = B_i + B_i\ell = \mathbb{C}(B_i,\mu_i)$ as before by the *-Necessity Proposition 3.9. By Witt's Theorem (which is applicable even in characteristic 2 since $n(x,y)$ is nonsingular) the fact that n, $\tilde{n}$ are equivalent and F_i an isometry from B_i to $\tilde{B}_i$ implies $B_i^{\perp}$ and $\tilde{B}_i^{\perp}$

are also isometric, so corresponding to ℓ_i we can find $\tilde{\ell}_i \in \tilde{B}_i^{\perp}$ with $\tilde{n}(\tilde{\ell}_i) = -\mu_i \neq 0$ and build $\tilde{B}_{i+1} = \tilde{B}_i + \tilde{B}_i\tilde{\ell}_i$ $= \mathbb{C}(\tilde{B}_i, \mu_i)$. Hence $F_{i+1}(c_i + d_i\ell_i) = F_i(c) + F_i(d)\tilde{\ell}_i$ defines a *-isomorphism $B_{i+1} = \mathbb{C}(B_i, \mu_i) \to \mathbb{C}(\tilde{B}_i, \mu_i) = \tilde{B}_{i+1}$ (since F_i is actually a *-isomorphism). This is the essential point - once ℓ and μ are given, the multiplication follows. Thus we can keep building up larger and larger isomorphic nonsingular subalgebras until eventually (by finite-dimensionality when $n(x,y) \not\equiv 0$) we have an isomorphism $A \to \tilde{A}$. ■

As an immediate consequence of the theorem we obtain our previous result that changing the parameter μ by a norm doesn't change the algebra $\mathbb{C}(B,\mu)$.

4.8 Lemma. If $n(u) \neq 0$ then $b \oplus (uc)\ell \xrightarrow{F} b \oplus c\ell$ is an isomorphism of $\mathbb{C}(B,\mu)$ with $\mathbb{C}(B,\mu n(u))$. Thus the isomorphism class of $\mathbb{C}(B,\mu)$ depends only on B and the coset of μ modulo the norm subgroup of Φ, i.e. the image of μ in $\Phi/n(B)$.

Proof. Clearly F is an isometry: $n(b + (uc)\ell) = n(b) - \mu n(uc)$ $= n(b) - \mu n(u)n(c) = \tilde{n}(b + c\ell)$. Thus by the Isomorphism Theorem (B,μ) and $(B,\mu n(u))$ are isomorphic.

In this case we actually saw directly F is an isomorphism in the Scaling Proposition I.1.12. ■

An another consequence we can establish the fact, alluded to in Chapter I, that all isotopes of a Cayley algebra are isomorphic. By the Isomorphism Theorem it suffices if they have equivalent norm forms. But in the Isotope Formula 1.20 we saw that the isotope

$\mathbb{C}^{(u,v)}$ had norm form $n^{(u,v)}(x) = n(uv)n(x)$, so the bijection L_{uv}: $\mathbb{C}^{(u,v)} \to \mathbb{C}$ satisfies $n(L_{uv}x) = n(uv \cdot x) = n(uv)n(x) = n^{(u,v)}(x)$. Thus L_{uv} is an equivalence and the isotope is isomorphic to $\mathbb{C}$.

4.9 (Isotopy Theorem for Cayley Algebras) All isotopes $\mathbb{C}^{(u,v)}$ of a Cayley algebra $\mathbb{C}$ are isomorphic to $\mathbb{C}$. ∎

Properties of Composition algebras

We can extract more information from the proof of the Hurwitz Theorem. Once more (compare 3.14)

4.10 (Nonsingularity Lemma). The bilinear norm form $n(x,y) = t(x\bar{y})$ of an extra-ordinary composition algebra (including Type I in characteristic 2) vanishes identically, and $x \to n(x)$ is a ring isomorphism of Ω into Φ. Otherwise the bilinear norm form of an ordinary composition algebra is nonsingular. ∎

4.11 (Symmetric Lemma) The symmetric elements of a composition algebra are precisely the scalars $\Phi 1$ (except for Types 0, III, IV in characteristic 2).

Proof. If $x = a + b\ell$ in $\mathbb{C}(B,\mu)$ coincides with $x^* = \bar{a} - b\ell$ then $a = \bar{a}$ and (in characteristic $\neq 2$) $b\ell = 0$, so $x = a$ is symmetric in B. Starting with $\Phi 1$ the symmetric elements never grow.

In characteristic 2 the result holds trivially for $\Phi 1$ of Type I, and for $\Phi u + \Phi\bar{u}$ of Type II the only elements $x = \alpha u + \beta\bar{u}$

which are symmetric, $x = \bar{x} = \alpha\bar{u} + \beta u$, are the $x = \alpha u + \alpha\bar{u}$ $= \alpha 1$. ∎

On several occasions we will need invertible elements of various forms.

4.12 (Invertible Basis Lemma) A composition algebra has a basis of invertible elements, except for a split $\Phi e \boxplus \Phi e^*$ of Type II over $\Phi = \mathbb{Z}_2$.

Proof. Certainly any field of Type 0 or I has a basis of invertible elements, and whenever B has invertible basis $\{b_i\}$ then $\mathbb{C}(B,\mu)$ has invertible basis $\{b_i\}$, $\{b_i\ell\}$. In particular, Types II, III, IV have invertible bases in characteristic $\neq 2$.

We next consider the modified $B_2' = \Phi 1 + \Phi u$ of Type II in characteristic 2. If $n(u) = -\mu_1 \neq 0$ then $\{1,u\}$ is an invertible bases; if $\mu_1 = 0$ but there is a scalar $\lambda \neq 0, 1$ then $n(1-\lambda u)$ $= 1 - \lambda t(u) + \lambda n(u) = 1 - \lambda \neq 0$ so $\{1,1-\lambda u\}$ is an invertible basis. The only case where B_2' doesn't have an invertible basis is when $\mu_1 = 0$ and $\Phi = \mathbb{Z}_2$ contains only 0 and 1; in this case $B_2' = \mathbb{Z}_2 u \boxplus \mathbb{Z}_2 \bar{u}$ where $u, \bar{u} = 1-u$ are orthogonal idempotents ($u^2 = t(u)u - n(u)1 = u$), and in $B_2' = \{0,1,u,\bar{u}\}$ only the element 1 is invertible.

Even though B_2' has no invertible bases in the case $\Phi = \mathbb{Z}_2$, $\mu_1 = 0$, $B_3' = \mathbb{C}(B_2',\mu_2) = B_2' + B_2'j$ has invertible basis $\{1, j, u + j, 1 + uj\}$ since $n(1) = 1$, $n(j) = -\mu_2$, $n(u + j)$ $= n(u) - \mu_2 = -\mu_2$, $n(1 + uj) = 1 - \mu_2 n(u) = 1$. Then B_4 too has such a basis. Thus Types III and IV have invertible bases in all cases. ∎

4.13 (Invertible Supply Lemma) Any composition algebra of Type II-IV contains invertible skew elements $x - \bar{x}$, and any algebra of Type III-IV contains invertible commutators.

Proof. In characteristic $\neq 2$ any type II $B_2 = \Phi + \Phi i$ has invertible $i - \bar{i} = 2i$, while in characteristic 2 $B_2' = \Phi 1 + \Phi u$ has invertible $u - \bar{u} = u + \bar{u} = 1$. Any Type III-IV contains a Type II subalgebra, hence contains invertible $x - \bar{x}$.

Any Type III-IV algebra has the form $\mathbb{C}(B, \mu)$ for B of Type II-III: since B contains invertible $b - \bar{b}$ by the above, $\mathbb{C}(B, \mu)$ contains invertible $[b, \ell] = b\ell - \ell b = (b - \bar{b})\ell$. ∎

Cayley algebras exhibit the peculiar property of simple alternative-but-not-associative algebras that the nucleus and center coincide.

4.14 (Nucleus and Center Criterion) The nucleus $N(A)$, center $C(A)$, and $*$-center $C(A,*)$ of a composition algebra A are:

Type 0: $N(A) = C(A) = C(A,*) = A > \Phi 1$

Type I: $N(A) = C(A) = C(A,*) = A = \Phi 1$

Type II: $N(A) = C(A) = A$, $C(A,*) = \Phi 1$

Type III: $N(A) = A$, $C(A) = C(A,*) = \Phi 1$

Type IV: $N(A) = C(A) = C(A,*) = \Phi 1$.

In a Cayley algebra any element which merely commutes with A is a scalar.

Proof. The results for the commutative associative Types 0-II are clear (recalling all symmetric elements belong to

$\Phi 1$ in Type II). It is well-known that quaternion algebras are centralsimple associative algebras. Alternately, if $z = c+dj$ lies in the center of $\mathbb{C}(B_2,\mu_2) = B_2 + B_2 j$ then $[z,j] = [z,a] = 0$ for all $a \in B_2$ implies $(c-\bar{c})j + (d-\bar{d})j^2 = d(\bar{a}-a) = 0$. By 4.13 there exist invertible $\bar{a} - a$, so $d = 0$, and by 4.11 $c = \bar{c}$ in B_2 implies $z = c = \gamma 1 \in \Phi 1$.

If $z = c + d\ell$ commutes with everything in $\mathbb{C}(B_3,\mu_3) = B_3 + B_3\ell$ then $[z,\ell] = [z,a] = 0$ for all $a \in B_3$ implies $(c-\bar{c})\ell + \mu(d-\bar{d}) = [c,a] + \{d(\bar{a}-a)\}\ell = 0$. By the above $[c,B_3] = 0$ implies $c = \gamma 1$, and $d(\bar{a}-a) = 0$ implies $d = 0$ since by 4.13 again there exist invertible $\bar{a} - a$, so $z = \gamma 1 \in \Phi 1$.

If z associates with everything in $\mathbb{C}(B_3,\mu_3)$ then $[a,b,z] = [a,z,\ell] = 0$ for all $a,b \in B_3$. Thus $\{d(ab-ba)\}\ell = (ac-ca)\ell + \mu(da-ad) = 0$, so $d[a,b] = [a,c] = [a,d] = 0$ for all a,b. By 4.13 there exist invertible $[a,b]$, so $d = 0$, and by the above $[c,B_3] = 0$ implies $c = \gamma 1$, so $z = \gamma 1 \in \Phi 1$. ∎

4.15 Corollary. If an element of a composition algebra commutes with everything, it also associates with everything. ∎

4.16 Corollary. A composition algebra is either associative, or else every element which associates with everything also commutes with everything. ∎

We can also show that all Cayley algebras are simple, indeed contain no proper one-sided ideals whatsoever.

4.17 (One-Sided Ideal Proposition) A Cayley algebra contains no proper one-sided ideals.

Proof. Because of the symmetry resulting from the involution, it suffices to show there are no proper left ideals B in a Cayley algebra $\mathbb{C} = \mathbb{C}(A, \mu)$ (A a quaternion algebra). If B is nonzero it contains an element $c + b\ell$ for $b \neq 0$ (if $c + 0\ell \in B$ then also $\ell(c + 0\ell) = \ell c = \bar{c}\ell \in B$ for $\bar{c} \neq 0$); since it is a left ideal, B also contains $z[x,y,c+b\ell] = z[x,y,c] + \{[b(xy) - (by)x]z\}\ell = \{b[x,y]z\}\ell$ for all $x,y,z \in A$. Thus B contains $\{b[A,A]A\}\ell = \{bA\}\ell$ ($[A,A]A$ is an ideal in the simple unital quaternion algebra A). So far we have left-multiplied B only by A. If we now left-multiply by $A\ell$ we see B contains $A\ell\{(bA)\ell\} = \mu(\overline{Ab})A = A\bar{b}A$. Again by simplicity, $\bar{b} \neq 0$ implies B contains A, therefore $\ell \cdot A = \bar{A}\ell = A\ell$ as well, and $B = A + A\ell = \mathbb{C}$. Thus as soon as $B \neq 0$ we have $B = \mathbb{C}$. ∎

4.18 Corollary. A Cayley algebra is a simple alternative algebra of dimension 8. ∎

Though they have no proper <u>one-sided ideals</u>, Cayley algebras need not be division algebras (witness the split Cayley algebras). As we mentioned in Chapter I, an alternative algebra is a division algebra iff it has no proper <u>inner ideals</u>.

Recall that an algebra with involution is ∗-simple if it is not trivial and has no proper ∗-ideals (ideals invariant under the involution).

4.19 (Simplicity Theorem) Any composition algebra over a field Φ is $*$-simple, and all are simple except for the case of a split 2-simensional composition algebra $\Phi e \boxplus \Phi e^*$.

Proof. Note that $*$-simplicity is easy to prove: the Cayley-Dickson Formula (3.3) shows that if C is a proper $*$-ideal in B then $C + C\ell$ is a proper $*$-ideal in $\mathbb{C}(B,\mu)$, so that if at any stage of the Cayley-Dickson process there were a proper $*$-ideal there would remain one at the final 8-dimensional stage, whereas a Cayley algebra hasn't even got proper one-sided ideals by 4.17. [Or directly: if C is a proper $*$-ideal then $t(c) = c + c^*$, $n(c) = cc^*$ cannot be invertible, so they are zero, and $n(C) = 0$, $n(C,A) = t(CA^*) \subset t(C) = 0$ by 1.12 implies $C = 0$ by nondegeneracy of n].

The inseparable extensions Ω (including $\Omega = \Phi$) are trivially simple, and by 4.18 the Cayley algebras are too. If A of dimension 2 or 4 has a proper ideal C then $n(C) = 0$ since $C \neq A$ has no invertible elements, yet $t(C) \supset t(C\bar{A}) = n(C,A) \neq 0$ by 1.12 since $C \neq 0$ and $n(x,y)$ is nonsingular. Thus there is a proper idempotent e in C; $t(e) = 1$ but $n(e) = 0$. If A has dimension 2 then $e + e^* = t(e) = 1$ and $ee^* = e^*e = n(e) = 0$ shows $A = \Phi e \boxplus \Phi e^*$ is a direct sum of two copies of Φ. If A has dimension 4 then $A \supset B = \Phi e \oplus \Phi e^*$, so $A = \mathbb{C}(B,\mu) = B + Bj$ for any $j \perp B$ by 4.2. Then $e \in C$ implies ej and $je = e^*j$ lie in C, which contradicts total isotropy of C: $n(ej, e^*j) = \mu n(e,e^*) = \mu \neq 0$. Therefore no such C exists in dimension 4. ∎

By means of the Inverse Criterion 1.17 for composition algebras we can decide when the Cayley-Dickson construction yields a division algebra.

4.20 (Division Algebra Construction) If B is a composition algebra then $\mathbb{C}(B,\mu)$ is a division algebra iff B is a Division algebra and $\mu \notin n(B)$ is not a norm.

Proof. The condition for $\mathbb{C}(B,\mu)$ to be a division algebra is the anisotropy condition that its norm not represent zero, $n(x) = n(b) - \mu n(c) \neq 0$ for $x = b + c\ell \neq 0$. Clearly this implies $n(b) \neq 0$ for $b \neq 0$ (take $c = 0$), so B must be a division algebra, and $n(b) \neq \mu$ (take $c = 1$), so μ is not a norm.

Conversely, if these two conditions are met then $n(b) - \mu n(c) \neq 0$ if $c = 0$ but $b \neq 0$ (since $n(b) \neq 0$) and $n(b) - \mu n(c) \neq 0$ if $c \neq 0$ (since $\mu \neq n(b)n(c)^{-1} = n(bc^{-1})$), so $n(x) \neq 0$ for $x \neq 0$. ∎

Split composition algebras

Idempotents in a composition algebra are easily characterized in terms of the trace and norm.

4.21 (Idempotent Criterion) An element e in a composition algebra is a proper idempotent iff $t(e) = 1$, $n(e) = 0$.

Proof. If $t(e) = 1$, $n(e) = 0$ then $e^2 = t(e)e - n(e)1 = e$ is idempotent; it is proper ($e \neq 1, 0$) since $t(e) = 1$ but $t(1) = 2$ and $t(0) = 0$. Conversely, if e is a proper idempotent then $0 = e^2 - t(e)e + n(e)1 = \{1-t(e)\}e + n(e)1$ implies $1 - t(e) = n(e) = 0$ since $e \neq \lambda 1$ ($\lambda 1$ is idempotent only for $\lambda = 1$ or $\lambda = 0$, and $e \neq 1$ or 0). ∎

As soon as the norm form represents zero, the whole composition algebra dissolves into a very simple form.

4.23 (Splitting Equivalence Theorem) The following conditions are equivalent for a composition algebra A over a field Φ.

(i) A is not a division algebra

(ii) A has zero divisors

(iii) the norm form is isotropic (represents zero), $n(x) = 0$ for some $x \neq 0$

(iv) A contains a proper idempotent $e \neq 1, 0$.

Proof. By Corollary 1.18 to the Inverse Theorem, (i) and (iii) are equivalent; clearly (ii) implies (i) (zero divisors $xy = 0$ destroy injectivity of L_x), and (iii) implies (ii) since if $n(x) = 0$ for $x \neq 0$ then $x\bar{x} = 0$ where $\bar{x} \neq 0$ (recall $\bar{\bar{x}} = x$!).

If $e \neq 1, 0$ is idempotent then $n(e) = 0$, and n represents zero nontrivially. Thus (iv) $\Longrightarrow$ (iii). Conversely, suppose $n(x) = 0$ for some $x \neq 0$. Since the norm form is nondegenerate, $n(x,A) \neq 0$, and we can find y with $n(x,\bar{y}) = 1$. Thus $e = xy$ has trace $t(e) = t(xy) = n(x,\bar{y}) = 1$ (see 1.12) and norm $n(e) = n(xy) = n(x)n(y) = 0$, so that e is a proper idempotent. Thus (iii) $\Longrightarrow$ (iv). ∎

We say a composition algebra is **split** if it contains a proper idempotent $e \neq 1,0$. By the above, a composition algebra is either split or a division algebra. It is very important that COMPOSITION ALGEBRAS COME IN TWO KINDS, DIVISION ALGEBRAS OR SPLIT ALGEBRAS (according as n does not or does represent zero). Another important fact is that ALL SPLIT COMPOSITION ALGEBRAS OF A GIVEN DIMENSION LOOK ALIKE.

4.23 (Split Isomorphism Theorem) Any two split composition algebras of the same dimension over the field Φ are isomorphic.

Proof. We will show that the norm form of a split algebra necessarily has maximal Witt index. Since any two quadratic forms of the same (even) dimension having maximal Witt idex are equivalent, and since equivalence of norm forms implies isomorphism of algebras by the Isomorphism Theorem, this will establish our result.

If A is split it contains an idempotent e with $t(e) = 1$, $n(e) = 0$, so $e + e^* = 1$ and $e^*e = 0$. Thus $A = A1 \subset Ae + Ae^*$ is a sum of two subspaces; this sum is direct because $ae = be^*$ implies $ae = ae^2 = (ae)e = (be^*)e = b(e^*e) = 0$. (This is a particular case of the Peirce Decomposition relative to two orthogonal idempotents discussed in Chapter VI.) Thus we have decomposed A into a direct sum $A = Ae \oplus Ae^*$ of totally isotropic subspaces (note $n(Ae) = n(A)n(e) = 0$, $n(Ae^*) = n(A)n(e^*) = n(A)n(e) = 0$) so that by definition it has maximal Witt index. ∎

Thus if you've seen one split algebra, you've seen them all. In particular, the split Cayley algebras introduced in Chapter I coincide with split Cayley algebras in our present sense. One way of building a split algebra is to take $\mathbb{C}(B,\mu)$ for $\mu = 1$ (whether B is split or not) because the element $x = 1 + \ell \neq 0$ has norm $n(x) = 1 - \mu = 0$. (In 1.00 we gave a proof that $\mathbb{C}(B,1)$ was split in characteristic $\neq 2$). Therefore we obtain the following (exhaustive) list of split composition algebras over a field Φ.

Dimension 4. We obtain a split four-dimensional algebra $\mathbb{C}(\Phi;1,1) = \{\Phi e + \Phi e^*\} \oplus \{\Phi e + \Phi e^*\}j = \Phi e_{11} + \Phi e_{22} + \Phi e_{12} + \Phi e_{21}$ which is isomorphic to the algebra $M_2(\Phi)$ of 2×2 matrices over Φ with standard involution $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \to \begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}$. Here $e_{11} = e, e_{22} = e^*, e_{12} = ej, e_{21} = e^*j$ act like matrix units. This can be checked directly, or note that we have an imbedding $\Phi e \oplus \Phi e^* \to \Phi e_{11} + \Phi e_{22}$ by $\alpha e + \delta e^* \to \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}$; since $j = e_{12} + e_{21} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ satisfies $jb = b^*j$ for all $b \in \Phi e_{11} + \Phi e_{22}$ and $j^2 = 1$, by 3.9 we have an isomorphism $\mathbb{C}(\Phi e \oplus \Phi e^*, 1) \to M_2(\Phi)$ via $(\alpha e + \delta e^*) + (\beta e + \gamma e^*)j \to (\alpha e_{11} + \delta e_{22}) + (\beta e_{11} + \gamma e_{22})(e_{12} + e_{21}) = \alpha e_{11} + \delta e_{22} + \beta e_{12} + \gamma e_{21} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$.

It might be well to stop for a moment and consider the involution in the (split) quaternion algebra $M_2(\Phi)$ in more detail. Here

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}^* = \begin{pmatrix}\alpha_{22} & -\alpha_{12}\\ -\alpha_{21} & \alpha_{11}\end{pmatrix}$$

Note that this is very different from the transpose involution $x \to x^t$ or any isotope $x \to cx^tc^{-1}$ thereof. The essential fact is that all traces $t(x) = x + x^*$ are scalars in $\Phi 1$. No other matrix algebra $M_n(\Phi)$, $n > 2$, can carry such an involution. Indeed, you will recall from the associative theory (the Cartan-Brauer-Hua Theorem, for example) that these quaternion algebras with standard involution are the unique exceptions to many general statements about associative algebras with involution.

The norm and trace are the usual ones for matrices $x = \begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$, Dimension 1. A one-dimensional composition algebra $\Phi 1$ is never split according to our definition.

Dimension 2. If the characteristic $\neq 2$ the split two-dimensional composition algebras look like $\mathbb{C}(\Phi,1) = \Phi 1 + \Phi i$ with $i^2 = 1$. Then $\mathbb{C} = \Phi e \boxplus \Phi e^*$ is a direct sum of two copies of Φ with exchange involution, where $e = \frac{1}{2}(1 + i)$, $e^* = \frac{1}{2}(1 - i)$. If the characteristic $= 2$ one takes $\mathbb{C}'(\Phi,0) = \Phi 1 + \Phi u$ where $t(u) = 1$, $n(u) = 0$ ($\mu = 0$ rather than $\mu = 1$) so again u is a proper idempotent and $\mathbb{C}' = \Phi u \boxplus \Phi u^*$ consists of two copies of Φ.

In $A = \Phi e_1 \boxplus \Phi e_2$ the trace and norm of an element $x = \alpha_1 e_1 + \alpha_2 e_2$ are $t(x) = \alpha_1 + \alpha_2$ and $n(x) = \alpha_1\alpha_2$. Notice the $n(x_1 \oplus x_2) = n(x_1)n(x_2)$ is the product of norms $n_i(\alpha e_i) = \alpha$ on Φe_i.

$n(x) = \det x = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$, $t(x) = \text{trace } x = \alpha_{11} + \alpha_{22}$.

Dimension 8. The unique eight-dimensional split Cayley algebra $\mathbb{C}(\Phi,1,1,1)$ is obtained as $M_2(\Phi) + M_2(\Phi)\ell = \{\Phi e_{11} + \Phi e_{22} + \Phi e_{12}^{(1)} + \Phi e_{21}^{(1)}\} + \{\Phi e_{12}^{(2)} + \Phi e_{12}^{(3)} + \Phi e_{21}^{(2)} + \Phi e_{21}^{(3)}\}$ for $e_{12}^{(1)} = e_{12}$, $e_{21}^{(1)} = e_{21}$, $e_{12}^{(2)} = e_{21}\ell$, $e_{21}^{(2)} = -e_{12}\ell$, $e_{12}^{(3)} = e_{11}\ell$, $e_{21}^{(3)} = e_{22}\ell$ where $e_{12}^{(m)}$, $e_{21}^{(m)}$ again act like matrix units e_{12}, e_{21}, but we have in addition products $e_{12}^{(m)} e_{21}^{(n)} = e_{21}^{(m)} e_{12}^{(n)} = 0$ if $m \neq n$ and $e_{12}^{(m)} e_{12}^{(m+1)} = e_{21}^{(m+2)}$, $e_{21}^{(m+1)} e_{21}^{(m)} = e_{12}^{(m+2)}$ (indices mod 3). (These are "Cayley matrix units" as in Section VI.5).

Although all split algebras of a given dimension look alike, the same is by no means true of division algebras. The classification of division algebras depends very much on arithmetic properties of the base field. We can say

(i) there are no composition division algebras of dimension > 1 over an algebraically closed field

(ii) there are no composition division algebras of dimension > 2 over a finite field

(iii) the only composition division algebra of dimension 2 over the real field $\mathbb{R}$ is the field $\mathbb{C} = \mathbb{C}(\mathbb{R},-1)$ of complex numbers; the only quaternion division algebra over the reals is the algebra of ordinary quaternions $\mathbb{C}(\mathbb{R},-1,-1)$; and the only Cayley division algebra over the reals is the algebra of ordinary Cayley numbers $\mathbb{C}(\mathbb{R},-1,-1,-1)$.

The algebraically closed case (i) is trivial since any non-constant form $n(x_1,\ldots,x_n)$ for $n > 1$ has nontrivial zeros in an algebraically closed field; similarly for a finite field Φ the norm form $n(x_1,\ldots,x_n)$ has more variables $n = 4, 8$ than its degree 2, hence (by the Artin-Chevalley Theorem) has a nontrivial zero. For the real case $\mathbb{R}$ we know that if we ever take $\mu = 1$ in the Cayley-Dickson process, $C(B,\mu)$ will be split; but that leaves us only $\mu = -1$ each time, because every real μ can be written $\mu = \pm\alpha^2 = \pm n(\alpha)$ (according as μ is positive or negative), so $\mathbb{C}(B,\mu) \cong \mathbb{C}(B,\pm 1)$ by Lemma 4.8.

VIII.4 Exercises

4.1 In the proof of the Simplicity Theorem 4.19, use $*$-simplicity of A to show that if C is a proper ideal in A then $A = C \boxplus C^*$. In dimension 2 show $C = \Phi c$ where $c^2 = \gamma c$ for $\gamma \neq 0$; conclude $C = \Phi e$.

4.2 Give an example of an isometry $A \overset{F}{\to} \tilde{A}$ of 4-dimensional composition algebras such that $F(1) = 1$ but F is not an isomorphism. Is this possible in dimension 2?

4.3 Give an example of an isomorphism $B \overset{F}{\to} \hat{B}$ of 4-dimensional unital subalgebras (singular, of course!) which cannot be extended to an isomorphism $A \to \tilde{A}$ of 8-dimensional composition algebras. Show that nevertheless A and $\tilde{A}$ must be isomorphic.

4.4 Describe all proper nilpotent elements in a composition algebra (Nilpotent Criterion).

VIII.4.1 Problem Set: Wright's Theorem on Absolute-Valued Algebras

An **absolute value** on an algebra A over the field $\mathbb{R}$ of real numbers is a real-valued function $|x|$ on A satisfying

(i) $|x| > 0$ for $x \neq 0$

(ii) $|\alpha x| = |\alpha||x|$ for $\alpha \in \mathbb{R}$

(iii) $|x+y| \leq |x| + |y|$

(iv) $|xy| = |x||y|$.

The last relation shows A cannot have any zero divisors.

1. If A is a composition division algebra over $\mathbb{R}$, show $|x| = \sqrt{n(x)}$ is a well-defined absolute value on A.

We want to establish the converse, that every absolute-valued (nonassociative) division algebra is a composition algebra with $|x| = \sqrt{n(x)}$. What we must do is show $n(x) = |x|^2$ is a nondegenerate quadratic form on A permitting composition. It certainly permits composition, $n(xy) = |xy|^2 = |x|^2|y|^2 = n(x)n(y)$ by (iv), and also $n(\alpha x) = \alpha^2 n(x)$ by (ii). The whole difficulty resides in showing n is quadratic, i.e. $n(x,y)$ is bilinear.

2. (Jordan-von Neumann Characterization of Inner Product Spaces) Show a function $f(x)$ on a Φ-module ($\frac{1}{2} \in \Phi$) which satisfies $f(\alpha x) = \alpha^2 f(x)$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ necessarily has the form $f(x) = g(x,x)$ where $g(x,y) = \frac{1}{4}\{f(x+y) - f(x-y)\}$ is a symmetric $\mathbb{Z}$-bilinear function. If $\Phi = \mathbb{R}$ and $f(x+y) \leq f(x) + f(y)$, show g is $\mathbb{R}$-bilinear.

3. Conclude that a normed linear space (or Banach space) is an inner product space (or Hilbert space) iff it satisfies the parallelogram

law $||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$. Conclude further a normed linear space X is an inner product space iff every 2-dimensional subspace X_o is. Observe that a 2-dimensional normed space is an inner product space iff its unit sphere $S = \{x \mid ||x|| = 1\}$ is an ellipse $\{x \mid g(x,x) = 1\}$, g a symmetric bilinear form.

4. Show that in a normed division algebra $|x+y|^2 + |x-y|^2 \geq 4$ if $|x| = |y| = 1$.

5. Show that if $|\ |$ is a norm on a 2-dimensional real vector space satisfying $|x+y|^2 + |x-y|^2 \geq 4$ for x,y in the unit sphere S, and $||\ ||$ a norm with unit sphere an ellipse E $(||x||^2 = g(x,x))$ such that E lies inside S with intersection $E \cap S$ consisting of more than 2 points, then necessarily E = S and $|\ | = ||\ ||$ is given by the inner product g.

6. Let $|\ |$ be a norm on a 2-dimensional space with unit sphere S. Show that if E is an ellipse $\{x \mid g(x,x) = 1\} = \{x \mid q(x) = 1\}$ (q quadratic form) contained inside S and having maximal area among such ellipses, then E meets S in at least 4 points.

7. Prove the <u>Day-Schoenberg Theorem</u>: If a normed linear space satisfies $|x+y|^2 + |x-y|^2 \geq 4$ for $|x| = |y| = 1$, it is an inner product space.

8. Prove <u>Wright's Theorem</u>: An absolute valued real division algebra is a composition algebra, hence is either $\mathbb{R}$, $\mathbb{C}$, Q, or $\mathcal{C}$ under $|x| = \sqrt{n(x)}$.

VII.4.2 Problem Set: Albert's Theorem on Absolute-Valued Algebras

By techniques of linear algebra (rather than geometry) one can establish the finite-dimensional version of Wright's Theorem.

1. Show that if L_x satisfies an equation of degree 2 for each x in a unital algebra A, then A is left alternative.

   Thus we will try to prove L_x is degree 2 when A is absolute-valued (a similar argument on R_x will give right alternativity). This leads us to investigate the minimum polynomial of L_x.

2. For a linear transformation T on a finite-dimensional space over $\mathbb{R}$ let $\sigma(T)$, the **Spectrum** of T, consist of the characteristic roots of T (roots of $\det|\lambda 1 - T| = 0$); $\sigma(T)$ is a subset of $\mathbb{C}$. Show $\sigma(p(T)) = p(\sigma(T))$ for any polynomial $p(\lambda)$.
3. Show that if X is a finite-dimensional normed space and T is a bounded linear transformation, $|Tx| \leq |T|\ |x|$ for all x, then $|\sigma(T)| \leq |T|$ (i.e. the spectrum is contained in a disc of radius $|T|$).
4. If T, T^{-1} are bounded and $|T^{-1}| = |T|^{-1}$ (for example, if $|Tx| = \tau|x|$ for all x then $|T| = \tau$ and $|T^{-1}| = \tau^{-1}$) show $|\sigma(T)| = |T|$ (i.e. $\sigma(T)$ lies on the circle of radius $|T|$).
5. If A is an absolute-valued finite-dimensional unital algebra, show that any L_x has at most two (conjugate) characteristic roots.
6. Show that if $S = I - Z$ for Z nilpotent has $|Sx| = |x|$ for all x, then $Z = 0$ and $S = I$.
7. Show that any L_x for x in a unital finite-dimensional absolute-valued algebra has degree 2.

8. Show every finite-dimensional absolute-valued algebra has a unital isotope.

9. Prove <u>Albert's Theorem</u>: Every finite-dimensional absolute-valued algebra over $\mathbb{R}$ is an isotope of $\mathbb{R}$, $\mathbb{C}$, Q, or $\mathcal{K}$ (and every unital one is itself one of $\mathbb{R}$, $\mathbb{C}$, Q, or $\mathcal{K}$).