§3. The Cayley-Dickson process

The Cayley-Dickson construction builds commutative associative algebras with nontrivial involution out of commutative associative algebras with trivial involution, noncommutative associative algebras out of commutative associative algebras with nontrivial involution, nonassociative alternative algebras out of noncommutative associative algebras, and nonalternative algebras out of nonassociative algebras. By means of this construction the Cayley-Dickson process starts from the base field and successively builds composition algebras of dimension 1,2,4 and 8 (the base field, quadratic extensions, quaternion algebras, and Cayley algebras respectively).

The Cayley-Dickson process, as its name suggests, is due to A. A. Albert. It is a process for starting with the base field and successively building up larger and larger composition algebras. At each stage we double the previous algebra by the Cayley-Dickson construction. Let us recall from Chapter I how that construction goes (always over a field Φ).

The Construction

Given a nonzero scalar $\mu \in \Phi$ and a unital algebra B with scalar involution $b \to \bar{b}$ (so $b\bar{b} = \bar{b}b = n(b)$ and $t(b) = b + \bar{b}$

lie in $\Phi 1$), we build a new algebra

(3.1) $$\mathbb{C}(B,\mu) = B \oplus B\ell$$

from two copies of B as linear space, with new involution

(3.2) $$(b+c\ell)^* = \bar{b} - c\ell$$

and new multiplication given as in I. 1.8 by the Cayley-Dickson Formula

(3.3) $$(b_1+c_1\ell)(b_2+c_2\ell) = (b_1b_2+\mu\bar{c}_2c_1) + (c_2b_1+c_1\bar{b}_2)\ell .$$

This eminently forgettable formula can be broken down into bite-sized pieces. Besides the fact that B is imbedded as a subalgebra with its usual multiplication we have

(3.4) $$b\ell = \ell\bar{b}$$

(3.5) $$b(c\ell) = (cb)\ell$$

(3.6) $$(c\ell)b = (c\bar{b})\ell$$

(3.7) $$(b\ell)(c\ell) = \mu\bar{c}b .$$

Some helpful mnemonic devices: in (3.5) notice that to multiply b against $c\ell$ you slip the b in behind the c, and also in (3.6) the b gets put between the c and the ℓ, but in moving past the ℓ it gets conjugated as in (3.4). (WARNING: many authors use $B \oplus \ell B$ instead, which turns all these formulas aound.)

The multiplication formula is forced upon us: we <u>must</u> define multiplication by (3.3) if we want to have $\ell^2 = \mu 1$ and $b\ell = \ell\bar{b}$. Recall that we are now working over a field Φ, so $\mu \neq 0$ implies μ is invertible.

3.8 (Necessity Proposition) If B is a unital subalgebra with scalar involution of a unital alternative algebra A, and ℓ an element of A satisfying $\ell^2 = \mu 1 \neq 0$ and $b\ell = \ell\bar{b}$ for all $b \in B$, then $B + B\ell$ is a subalgebra of A whose multiplication is given by the Cayley-Dickson formula, and the map $b \oplus c\ell \to b + c\ell$ is a homomorphism of $\mathbb{C}(B,\mu)$ onto the subalgebra $B + B\ell$. When $B \cap B\ell = 0$, the map is an isomorphism.

Proof. Since (3.4) is one of our assumptions, we have $(b\ell)(c\ell) = (\ell\bar{b})(c\ell) = \ell(\bar{b}c)\ell$ (Middle Moufang) $= \ell\{\ell\overline{(\bar{b}c)}\} = \ell^2(\bar{c}b) = \mu\,\bar{c}b$, so (3.7) is forced upon us. From alternativity $(cb)\ell + (c\ell)b = c(b\ell+\ell b)$ we see $(c\ell)b = c\{(b+\bar{b})\ell\} - (cb)\ell$ (by (3.4)) $= \{c(b+\bar{b}) - cb\}\ell$ (since $b + \bar{b} \in \Phi 1$) $= (c\bar{b})\ell$, so (3.6) too is forced. If A had an involution with $\bar{\ell} = -\ell$ this would imply (3.5), but since we are not assuming an involution we argue dually: $b(c\ell) = b(\ell\bar{c}) = -\ell(b\bar{c}) + (b\ell+\ell b)\bar{c} = -\ell(b\bar{c}) + \{\ell(b+\bar{b})\}\bar{c} = \ell\{-b\bar{c} + (b+\bar{b})\bar{c}\} = \ell(\bar{b}\bar{c}) = \ell(\overline{cb}) = (cb)\ell$.

We could also derive (3.5) - (3.6) from the Moufang formulas: for example, $\mu\{b(c\ell)\} = \ell(\ell\{b(\ell\bar{c})\}) = \ell(\{\ell b\ell\}\bar{c}) = \ell(\{\ell^2\bar{b}\}\bar{c}) = \mu\ell(\overline{bc}) = \mu(cb)\ell$ and $\mu\{(c\ell)b\} = ([(c\ell)b]\ell)\ell =$

$(c\{\ell b\ell\})\ell = \mu(c\bar{b})\ell$.

Thus the various pieces (3.4) - (3.7) of the Cayley-Dickson formula are forced upon us. In particular, $B + B\ell$ is a subalgebra.

Since both $\mathbb{C}(B,\mu)$ and $B + B\ell$ have multiplication given by the Cayley-Dickson formula, the map $b \oplus c\ell \to b + c\ell$ is a homomorphism. If $B \cap B\ell = 0$ then $b + c\ell = 0 \Rightarrow b = c\ell = 0$, and $c\ell = 0 \Rightarrow \mu c = (c\ell)\ell = 0 \Rightarrow c = 0$, so in these cases the map is injective. ∎

When A itself has an involution the proposition takes the form

3.9 (*-Necessity Proposition) If B is a unital subalgebra of an alternative algebra A with scalar involution, and $\ell \in B^{\perp}$ has norm $n(\ell) = -\mu \neq 0$ then $B + B\ell$ is a *-subalgebra of A with Cayley-Dickson multiplication and involution, and the map $b \oplus c\ell \to b + c\ell$ is a *-homomorphism of $\mathbb{C}(B,\mu)$ onto $B + B\ell$. If the norm bilinear form is nonsingular on B then the map is a *-isomorphism.

Proof. If $\ell \in B$ then $t(b\ell) = n(b^*,\ell) \in n(B,B^{\perp}) = 0$ by (1.6) (note that a unital subalgebra is automatically a *-subalgebra since $b^* = t(b)1 - b$), so $t(\ell) = t(b\ell) = 0$ imply $\ell + \ell^* = b\ell + \ell^*b^* = 0$, or $\ell^* = -\ell$ and $b\ell = \ell\bar{b}$ for all $b \in B$. Thus $\ell^2 = -\ell\ell^* = -n(\ell)1 = \mu 1$. By the previous proposition $b \oplus c\ell \to b + c\ell$ is a homomorphism of

$\mathbb{C}(B,\mu)$ onto the subalgebra $B + B\ell$. Since $(b+c\ell)^* = \bar{b} - c\ell$ as in (3.2), the map is a $*$-homomorphism.

If the norm bilinear form is nonsingular on B, $B \cap B^{\perp} = 0$, then $B \cap B\ell = 0$ because $B\ell \subset B^{\perp}$: $n(B\ell,B) = n(\ell,B^*B)$ (by (1.15)) $\subset n(B^{\perp},B) = 0$ since B is a $*$-subalgebra. Thus by the criterion in 3.8 the map is a $*$-isomorphism. ∎

This result will be crucial in the next section; it allows us to build up Cayley-Dickson algebras inside A because it says that under suitable conditions $B \subset A \Rightarrow \mathbb{C}(B,\mu) \subset A$.

Returning to $\mathbb{C}(B,\mu)$ in general, recall that $*$ is again a scalar involution with

$$\text{(3.10)} \qquad \begin{aligned} t(x) &= t(b) \\ n(x) &= n(b) - \mu n(c) \end{aligned} \qquad (x=b+c\ell)$$

(we established this for general Φ I, 1.10).

Notice that since it possesses a scalar involution, $\mathbb{C}(B,\mu)$ is of degree 2.

From the expression for the norm on $\mathbb{C}(B,\mu)$, we see that it inherits nondegeneracy from B:

3.11 (Nondegeneracy Criterion) If the norm bilinear form is nonsingular on B, it remains nonsingular as a bilinear form on $\mathbb{C}(B,\mu)$. If the norm bilinear form is singular on an alternative B but the norm quadratic form is nondegenerate, and if μ is not a norm, $\mu \neq n(b)$, then n is ... form on $\mathbb{C}(B,\mu)$

Proof. If $z = b + c\ell$ is in the radical of the norm bilinear form, $n(z,x) = 0$ for all $x = a + d\ell$, then from linearized (3.10) $n(b,a) - \mu n(c,d) = 0$ for all a,d; setting successively $d = 0$, $a = 0$ we see $n(b,B) = \mu n(c,B) = 0$. If $n(x,y)$ is nonsingular on B, this yields $b = c = 0$ and $z = 0$ so $n(x,y)$ is nonsingular on $\mathfrak{C}(B,\mu)$.

If we only assume the quadratic form is nondegenerate on B, then if $z \neq 0$ belongs to the radical of the norm form, $n(z) = n(z,x) = 0$ for all x, we have $n(b) = \mu n(c)$ yet $n(b,B) = n(c,B) = 0$, hence $n(b), n(c) \neq 0$ by nondegeneracy on B. But by (1.16) the norm in an alternative algebra permits composition, therefore $\mu = n(b)n(c)^{-1} = n(bc^{-1})$, contrary to our hypotheses that μ not be a norm. Consequently n is nondegenerate on $\mathfrak{C}(B,\mu)$. ∎

To see what sort of algebras the Cayley-Dickson construction leads to, we prove

3.12 (Criterion) If B is a unital algebra with scalar involution then (1) $\mathfrak{C}(B,\mu)$ is commutative iff B is commutative with trivial involution; (2) $\mathfrak{C}(B,\mu)$ is associative iff B is commutative and associative; (3) $\mathfrak{C}(B,\mu)$ is alternative iff B is associative.

Proof. The commutators $[x,z]$ and associators $[x,y,z]$ for $x = a + b\ell$, $z = c + d\ell$ in $\mathfrak{C}(B,\mu)$ are spanned by elements of the form

(C1) $[a,c] = [a,c]$

(C2) $[a,d\ell] = \{d(a-\bar{a})\}\ell$

(C3) $[b\ell,d\ell] = \mu\{\bar{d}b-\bar{b}d\}$

(A1) $[a,b,c] = [a,b,c]$

(A2) $[a,b,d\ell] = \{d(ab)-(db)a\}\ell = \{d[a,b]-[d,b,a]\}\ell$

(A3) $[a,b\ell,c] = \{(ba)\bar{c}-(b\bar{c})a\}\ell = \{[b,a,\bar{c}]-[b,\bar{c},a]+b[a,\bar{c}]\}\ell$

(A4) $[b\ell,\bar{a},c] = \{(ba)\bar{c}-b(\bar{c}a)\}\ell = \{[b,a,\bar{c}]+b[a,\bar{c}]\}\ell$

(A5) $[a,b\ell,d\ell] = \mu\{\bar{d}(ba)-a(\bar{d}b)\} = \mu\{-[\bar{d},b,a]+[\bar{d}b,a]\}$

(A6) $[b\ell,\bar{a},d\ell] = \mu\{\bar{d}(ba)-(a\bar{d})b\} = \mu\{-[\bar{d},b,a]+[\bar{d}b,a]-[a,\bar{d},b]\}$

(A7) $[a\ell,b\ell,c] = \mu\{(\bar{b}a)c-(c\bar{b})a\} = \mu\{[\bar{b}a,c]-[c,\bar{b},a]\}$

(A8) $[a\ell,b\ell,d\ell] = \mu\{d(\bar{b}a)-a(\bar{b}d)\}\ell = \mu\{[d,\bar{b}a]+[\bar{b},a]d+[a,\bar{b},d]\}\ell$

where $a,b,c,d \in B$.

If $\mathbb{C}(B,\mu)$ is commutative so is the subalgebra B, and by (C2) $[a,\ell] = (a-\bar{a})\ell = 0$ implies the involution is trivial. Conversely, if $ab = ba$ and $\bar{a} = a$ for all a,b then (C1)-(C3) vanish and $\mathbb{C}(B,\mu)$ is commutative.

If $\mathbb{C}(B,\mu)$ is associative so is the subalgebra B, and by (A2) $[a,b,\ell] = [a,b]\ell = 0$ implies all $[a,b] = 0$ and B is commutative. Conversely, if B is commutative associative all commutators and associators are zero, so (A1)-(A8) vanish and $\mathbb{C}(B,\mu)$ is associative.

If $\mathbb{C}(B,\mu)$ is alternative then $0 = [a,b\ell,c]+[b\ell,a,c]$ $= -[a,b\ell,\bar{c}]+[b\ell,\bar{a},\bar{c}]$ ($x+\bar{x} = t(x)1$ is nuclear) $= [b,c,a]\ell$ for all a,b,c by (A3),(A4), so all $[b,c,a] = 0$ and B is associative.

Conversely, if B is associative we have left alternativity $[x,x,y] = 0$ for all $x = a+b\ell$, $y = c+d\ell$ since

$$[a,a,c+d\ell] = [a,a,c]+\{d[a,a]-[d,a,a]\}\ell \quad \text{(A1-2)}$$

$$[b\ell,b\ell,c+d\ell] = \mu\{[\bar{b}b,c]-[c,\bar{b},b]\} + \mu\{[d,\bar{b}b]+[\bar{b},b]d+[b,\bar{b},d]\}\ell \quad \text{(A7-8)}$$

$$[a,b\ell,c+d\ell]+[b\ell,a,c+d\ell] = \{[a,b\ell,c]-[b\ell,\bar{a},c]\}+\{[a,b\ell,d\ell]-[b\ell,\bar{a},d\ell]\} = -[b,\bar{c},a]\ell+\mu[a,\bar{d},b] \quad \text{(A3-4,A5-6)}$$

all vanish since all associators in B vanish and $\bar{b}b = b\bar{b} \in \Phi 1$ commutes with any c or d. Therefore $\mathbb{C}(B,\mu)$ is left alternative, and dually (via the involution) is right alternative. ∎

The Process

Thus you must start with an associative algebra in order to obtain an alternative one by the Cayley-Dickson construction; once you have built an alternative but not associative algebra you can go no further, anything further will not be alternative.

We are now in a position to build composition algebras by a method called the Cayley-Dickson process. We begin with B_1 being just the ring $\Phi 1$ of scalars with identity involution. Next we form a two dimensional $B_2 = \mathbb{C}(B_1, \mu_1) = \Phi 1 \oplus \Phi i$, which will be commutative associative since B_1 is commutative associative with identity involution; if Φ has characteristic $\neq 2$, the involution $\alpha + \beta i \to \alpha - \beta i$ is not the identity.

In characteristic 2 the usual process applied to any algebra with identity involution will still have identity involution. To break out of this cycle we must modify the process slightly. We form

$$B_2' = \Phi 1 \oplus \Phi u \qquad (u^2 - u + \mu_1 1 = 0)$$

where μ_1 is now <u>arbitrary</u> in Φ, so that B_2' is a commutative associative degree 2 algebra whose standard involution is determined by

$$u + \bar{u} = 1$$

and hence is nontrivial. Note $t(u) = 1$, $n(u) = \mu_1$. We will call this algebra $\mathbb{C}'(B_1, \mu_1)$ and pretend it is obtained by the Cayley-Dickson process. Remember: THE SECOND STAGE OF THE CAYLEY DICKSON PROCESS IS DEFINED DIFFERENTLY IN CHARACTERISTIC 2.

Once we have arrived at a second-stage algebra B_2 of dimension 2 with nontrivial scalar involution, we can form a 4-dimensional $B_3 = \mathbb{C}(B_2, \mu_2) = B_2 \oplus B_2 j$ which will be associative but (since B_2 has nontrivial involution) not commutative. These are precisely the quaternion algebras over Φ.

From B_3 we construct an 8-dimensional algebra $B_4 = \mathbb{C}(B_3, \mu_3)$ which will be alternative but (since B_3 is not commutative) not associative. Such an 8-dimensional algebra is called a Cayley algebra (alias Cayley-Dickson, Cayley-Graves, Albert-Dickson ; in analogy with quaternions, they are often called octaves or octonions). Thus Cayley algebras are those obtained from quaternion algebras by the Cayley-Dickson construction.

At this point the process stops, for further algebras will not be alternative (since B_4 is not associative).

These algebras

$$B_1 = \Phi 1,\ B_2 = \mathbb{C}(B_1, \mu_1),\ B_3 = \mathbb{C}(B_2, \mu_2),\ B_4 = \mathbb{C}(B_3, \mu_3)$$

of dimensions 1, 2, 4, 8 are the basic algebras obtained by the Cyaley-Dickson process, and so we will call them the Cayley- Dickson process algebras. If we had let the unmodified construction run on in characteristic 2 we would simply have gotten larger and larger commutative, associative algebras with identity involution such that $x^2 \in \Phi 1$ for all x, i.e. purely inseparable "extensions" (though not necessarily field extensions) of exponent 1.

Note that we use the term Cayley-Dickson construction for the general doubling process (3.1), applicable to any algebra with scalar involution, while the term Cayley-Dickson process is used for the particular case where we start the doubling process with the base field (and stop when the algebras ...

3.13 (Composition Proposition) All Cayley-Dickson process algebras are composition algebras: any algebra of dimension 1, 2, 4, or 8 obtained by the Cayley-Dickson process with nonzero parameters μ_1, μ_2, μ_3 has nondegenerate norm form. The algebras of dimension 2, 4, or 8 have nonsingular norm bilinear form $n(x,y)$; singularity of $n(x,y)$ is possible only in characteristic 2 and dimension 1.

Proof. As alternative algebras with scalar involutions, the Cayley-Dickson process algebras are of degree 2 and their norm forms permit composition (see (1.16)). To be composition algebras according to our definition, the norms must be nondegenerate. In the case of B_1, $n(\alpha,\beta) = 2\alpha\beta$ is nonsingular iff Φ has characteristic $\neq 2$, and $n(\alpha) = \alpha^2$ is always nondegenerate. If Φ has characteristic $\neq 2$ then by the Nondegeneracy Criterion 3.11 the quadratic extension B_2, the quaternion algebra B_3, and the Cayley algebra B_4 will all have nonsingular forms $n(x,y)$, and so are composition algebras. If Φ has characteristic 2 the form $n(x,y)$ on the modified $B_2' = \Phi 1 + \Phi u$ is nonsingular (being hyperbolic, $n(\alpha 1 + \beta u, \alpha' 1 + \beta' u) = \alpha\beta' + \alpha'\beta$), so again B_2', B_3, B_4 are composition algebras with nonsingular norm bilinear forms. ∎

In the next section we will see that conversely all composition algebras are obtained by the Cayley-Dickson process.

Examples

In case Φ has characteristic $\neq 2$, the algebras we have built look like the following.

Dimension 1: Base field

$B_1 = \Phi 1 = \Phi$ is a commutative associative field with identity involution $\overline{1} = 1$ and norm $n(\alpha_1) = \alpha_1^2$.

Dimension 2: Quadratic extension

$B_2 = \Phi 1 \oplus \Phi i$ is a commutative associative algebra with nontrivial involution $\overline{i} = -i$, multiplication $i^2 = \mu_1 1$, and norm $n(\alpha_2 1 + \alpha_2 i) = \alpha_1^2 - \alpha_2^2 \mu_1$.

Dimension 4: Quaternion algebra

$B_3 = \Phi 1 \oplus \Phi i \oplus \Phi j \oplus \Phi k$ is a non-commutative associative algebra with involution $\overline{i} = -i$, $\overline{j} = -j$, $\overline{k} = -k$, norm

$$n(\alpha_1 1 + \alpha_2 i + \alpha_3 j + \alpha_4 k) = \alpha_1^2 - \alpha_2^2 \mu_1 - \alpha_3^2 \mu_2 + \alpha_4^2 \mu_1 \mu_2,$$

and multiplication

$$i^2 = \mu_1 1,\ j^2 = \mu_2 1,\ k^2 = -\mu_1 \mu_2 1;$$

$$ij = -ji = k,\ jk = -kj = -\mu_2 i,\ ki = -ik = -\mu_1 j.$$

Dimension 8: Cayley algebra

$B_4 = \Phi 1 \oplus \Phi i \oplus \Phi j \oplus \Phi k \oplus \Phi \ell \oplus \Phi i\ell \oplus \Phi j\ell \oplus \Phi k\ell$ is an alternative but not associative algebra with involution $\overline{i} = -i, \overline{j} = -j, \overline{k} = -k, \overline{\ell} = -\ell, \overline{(i\ell)} = -i\ell, \overline{(j\ell)} = -j\ell, \overline{(k\ell)} = -k\ell$, norm

$$n(\alpha_1 1 + \alpha_2 i + \alpha_3 j + \alpha_4 k + \alpha_5 \ell + \alpha_6 i\ell + \alpha_7 j\ell + \alpha_8 k\ell)$$
$$= \alpha_1^2 - \alpha_2^2 \mu_1 - \alpha_3^2 \mu_2 + \alpha_4^2 \mu_1 \mu_2 - \alpha_5^2 \mu_3 + \alpha_6^2 \mu_1 \mu_3 + \alpha_7^2 \mu_2 \mu_3 - \alpha_8^2 \mu_1 \mu_2 \mu_3$$

and multiplication

$$^2 = \mu_1 1, j^2 = \mu_2 1, k^2 = -\mu_1 \mu_2 1, \ell^2 = \mu_3 1, (i\ell)^2 = -\mu_1 \mu_3 1, (j\ell)^2 = -\mu_2 \mu_3, (k\ell)^2 = \mu_1 \mu_2 \mu_3 1$$

$$ij = k = -ji, jk = -\mu_2 i = -kj, ki = -\mu_1 j = -ik, \ell i = -i\ell, \ell j = -j\ell, \ell k = -k\ell;$$

$i(i\ell) = \mu_1\ell = -(i\ell)i, \quad j(j\ell) = \mu_2\ell = -(j\ell)j, \quad k(k\ell) = -\mu_1\mu_2\ell = -(k\ell)k;$

$i(j\ell) = -k\ell = -(j\ell)i, \quad j(k\ell) = \mu_2 i\ell = -(k\ell)j, \quad k(i\ell) = \mu_1 j\ell = -(i\ell)k;$

$j(i\ell) = k\ell = -(i\ell)j, \quad k(j\ell) = -\mu_2 i\ell = -(j\ell)k, \quad i(k\ell) = -\mu_1 j\ell = -(k\ell)i;$

$(i\ell)(j\ell) = \mu_3 k = -(j\ell)(i\ell), (j\ell)(k\ell) = -\mu_2\mu_3 i = -(k\ell)(j\ell), (k\ell)(i\ell) = -\mu_1\mu_3 j = -(i\ell)(k\ell);$

$(i\ell)\ell = \mu_3 i = -\ell(i\ell), \quad (j\ell)\ell = \mu_3 j = -\ell(j\ell), \quad (k\ell)\ell = \mu_3 k = -\ell(k\ell);$

The results in characteristic 2 are (even ?) less memorable, since in place of 1,i we have 1,u with $u^2 = u - \mu_2 1$. Multiplication in the Cayley algebra can be summarized in a multiplication table for the products xy:

x \ y	1	i	j	k	ℓ	$i\ell$	$j\ell$	$k\ell$
1	1	i	j	k	ℓ	$i\ell$	$j\ell$	$k\ell$
i	i	$\mu_1 1$	k	$-\mu_1 j$	$i\ell$	$\mu_1\ell$	$-k\ell$	$-\mu_1 j\ell$
j	j	$-k$	$\mu_2 1$	$-\mu_2 i$	$j\ell$	$k\ell$	$\mu_2\ell$	$\mu_2 i\ell$
k	k	$-\mu_1 j$	$\mu_2 i$	$-\mu_1\mu_2 1$	$k\ell$	$\mu_1 j\ell$	$-\mu_2 i\ell$	$-\mu_1\mu_2\ell$
ℓ	ℓ	$-i\ell$	$-j\ell$	$-k\ell$	$\mu_3 1$	$-\mu_3 i$	$-\mu_3 j$	$-\mu_3 k$
$i\ell$	$i\ell$	$-\mu_1\ell$	$-k\ell$	$-\mu_1 j\ell$	$\mu_3 i$	$-\mu_1\mu_3 1$	$\mu_3 k$	$\mu_1\mu_3 j$
$j\ell$	$j\ell$	$k\ell$	$-\mu_2\ell$	$\mu_2 i\ell$	$\mu_3 j$	$-\mu_3 k$	$-\mu_2\mu_3 1$	$-\mu_2\mu_3 i$
$k\ell$	$k\ell$	$\mu_1 j\ell$	$-\mu_2 i\ell$	$\mu_1\mu_2\ell$	$\mu_3 k$	$-\mu_1\mu_3 j$	$\mu_2\mu_3 i$	$\mu_1\mu_2\mu_3 1$

Staring at this table usually is NOT the best way to understand the structure of a Cayley algebra.

If $\mathfrak{F} = \mathbb{R}$ is the field of real numbers and $\mu_1 = \mu_2 = \mu_3 = -1$ then the quadratic extension B_2 is simply the field $\mathbb{C}$ of complex numbers with conjugation as involution, B_3 is the division algebra $\mathfrak{Q}$ of (ordinary) quaternions with standard involution, and B_4 is the algebra $\mathfrak{C}$ of Cayley numbers; it is an alternative division algebra. IT IS OFTEN HELPFUL TO THINK OF THE CAYLEY-DICKSON PROCESS ALGEBRAS AS GENERALIZATIONS OF

$$\mathbb{R}, \mathbb{C}, \mathfrak{Q}, \mathfrak{C}.$$

VII.3 Exercises

3.1 Find the Cayley-Dickson formula for multiplication in $B + \ell B$ if $\ell^2 = \mu 1$, $\ell b = \bar{b}\ell$.

3.2 Try to define a general "modified Cayley-Dickson process" $\mathbb{C}(B,\nu) = Bu \oplus B\bar{u}$ with $u + \bar{u} = 1$, $u^2 - u + \nu 1 = 0$.

3.3 In the "modified" construction $B_2' = \Phi 1 \oplus \Phi u$ in arbitrary characteristic, $u^2 - u + \nu 1 = 0$, show $n(x,y)$ is nonsingular iff $1 + 4\nu$ is nonzero. Conclude in characteristic 2 it is nonsingular no matter what ν is chosen.

3.4 In characteristic $\neq 2$ show $u = \frac{1}{2}(1 + i)$ in $B_1 = \mathbb{C}(\Phi,\mu_1) = \Phi 1 \oplus \Phi i$ has $u + \bar{u} = 1$, $u^2 - u + \nu_1 1 = 0$. What is $1 + 4\nu_1$?

3.5 If B is a simple algebra, over an arbitrary ring Φ of scalars, with involution which is associative but not commutative, and $\mu \in \Phi$ is invertible, show the algebra $\mathbb{C}(B,\mu)$ defined as in (3.1) with the Cayley-Dickson formula (3.3) has no proper ideals, in particular is simple. (Notice B is not assumed unital nor the involution scalar, so $\mathbb{C}(B,\mu)$ need not be alternative. Also the element ℓ need not exist in $\mathbb{C}$, so $b\ell$ cannot be interpreted as a product of b with ℓ.)

3.6 If B is a unital algebra over a ring Φ of scalars, with an involution which is not necessarily scalar, show for cancellable μ that $\mathbb{C}(B,\mu)$ is commutative iff B is commutative and $* = I$ (examine commutators); show $\mathbb{C}$ is associative iff B is commutative associative (examine associators); show $\mathbb{C}$ is alternative iff (i) B is alternative, (ii) all $n(b) = b\bar{b} = \bar{b}b$ commute with B, (iii) all $b + t(b)$ associate with B (examine associators). Show from (iii) that if B has no 3-torsion then B is associative, so * is a central involution. Thus to get an alternative $\mathbb{C}$ the involution has to be at least central anyway; regarding B as an algebra over its center Ω, this means * is scalar over Ω.

3.7 Show that Propositions 3.8 and 3.9 hold over arbitrary rings of scalars Φ as long as μ is cancellable ($\mu x = 0 \Rightarrow x = 0$). Show 3.11 holds if μ is cancellable and (in the second part) does not satisfy $\mu n(c) = n(b)$ for any $n(c), n(b) \neq 0$. Show 3.12 holds no matter what μ we choose.