Chapter VII

Composition Algebras

§1. Degree 2 algebras

Algebras of degree 2 play a large role in nonassociative algebra. Over a field other than $\mathbb{Z}_2$ they are just the algebras in which every element satisfies an equation of degree 2. In a flexible degree 2 algebra there is a standard scalar involution, and conversely algebras with scalar involutions are always of degree 2. In alternative degree 2 algebras we can find simple expressions for inverses, U-operators, and isotopes in terms of the standard involution.

Defining Conditions

Throughout this chapter we will consider only unital algebras over a field Φ, though much of what we do goes over for an arbitrary ring of scalars (see Problem Set VII. 1.1). The properties of the composition algebras will indicate and illuminate the properties of general alternative algebras.

We say a unital nonassociative algebra A over the field is of **degree 2 over** Φ if every element $x \in A$ satisfies an equation of degree 2

(1.1) $$x^2 - t(x)x + n(x)1 = 0 \qquad (t(x),\ n(x) \in \Phi)$$

(1.2) $$t(1) = 2,\ n(1) = 1$$

where the **trace** t is a linear function and the **norm** n a quadratic function of x. (Such algebras are usually called *quadratic* algebras, but for ulterior motives we shall refrain from using this term.)

These hypotheses guarantee A is strictly degree 2, i.e. every scalar extension A_Ω remains degree 2 over Ω. Indeed, the linearization process applied to (1.1) yields $\{(x+y)^2-x^2-y^2\} - \{t(x+y)(x+y) - t(x)x-t(y)y\} + \{n(x+y)-n(x)-n(y)\}1 = 0$ or

(1.1') $$x \circ y - t(x)y - t(y)x + n(x,y)1 = 0$$

(using the hypothesis $t(x+y) = t(x) + t(y)$ of linearity of t; here $n(x,y)$ is bilinear by the quadratic nature of n). Once (1.1) and its linearization (1.1') hold for the elements of A, they hold for all elements $x = \Sigma\omega_i a_i$ in A_Ω ($\omega_i \in \Omega$, $a_i \in A$):

$$\begin{aligned}
x^2 &= \Sigma\omega_i^2 a_i^2 + \sum_{i<j} \omega_i\omega_j\, a_i \circ a_j \\
&= \sum_i \omega_i^2\{t(a_i)a_i - n(a_i)1\} + \sum_{i<j} \omega_i\omega_j\{t(a_i)a_j + t(a_j)a_i - n(a_i,a_j)1\} \\
&= \{\sum_{i=j} + \sum_{i<j} + \sum_{i>j}\}\omega_i\omega_j\, t(a_i)a_j - \{\sum_i \omega_i^2\, n(a_i) + \sum_{i<j}\omega_i\omega_j\, n(a_i,a_j)\}1 \\
&= \sum_j\{\sum_i \omega_i t(a_i)\}\omega_j a_j - \{\sum_i \omega_i^2\, n(a_i) + \sum_{i<j}\omega_i\omega_j\, n(a_i,a_j)\}1 \\
&= t_\Omega(x)x - n_\Omega(x)1
\end{aligned}$$

for $t_\Omega(x) = \Sigma\, \omega_j t(a_j)$ the linear extension of t on A to $1 \otimes t$ on $\Omega \otimes A = A_\Omega$, and $n_\Omega(x) = \Sigma\, \omega_i^2 n(a_i) + \sum_{i<j} \omega_i \omega_j n(a_i, a_j)$ the quadratic extension of n on A to A_Ω.

1.3 Remark. Being of degree 2 depends very much on the ring of scalars. If $\Phi \supset \Phi_o$ then any Φ-algebra is by restriction also a Φ_o-algebra, but in general it won't be of degree 2 over Φ_o if it is degree 2 over Φ. The trouble is that for A to be quadratic over Φ_o all $t(x), n(x)$ must be scalars in Φ_o, not just in Φ. ∎

1.4 Remark. The conditions (1.2) are not automatic consequences of (1.1): if $A = \Phi 1$ has dimension 1 then (1.1) holds for $t'(\lambda) = 2\lambda + \varepsilon\lambda$, $n'(\lambda) = \lambda^2 + \varepsilon\lambda^2$ but $t'(1) = 2 + \varepsilon$, $n'(1) = 1 + \varepsilon$ where ε can be arbitrary.

However, as soon as A has dimension 2 or more then (1.2) does follow automatically: setting $y = 1$ in (1.1') gives $2x = x \circ 1 = t(1)x + t(x)1 - n(x,1)1$, or

$$\{t(1) - 2\}x + \{t(x) - n(x,1)\}1 = 0$$

so if there exists $x \notin \Phi 1$ both coefficients must vanish by linear independence, $t(1) = 2$ and $t(x) = n(x,1)$. Then $0 = 1^2 - t(1)1 + n(1)1 = \{n(1) - 1\}1$ shows $n(1) = 1$.

Thus (1.2) is automatic if $A > \Phi 1$, and the effect of (1.2) is merely to insure that we have chosen the proper trace and norm when $A = \Phi 1$. From our point of view, a t and n which don't satisfy (1.2) are not really the trace and norm for A. ∎

1.5 Remark. In the course of the previous remark we incidentally deduced

$$t(x) = n(x,1) \tag{1.6}$$

from (1.1) when $x \notin \Phi 1$. When $x = \alpha 1 \in \Phi 1$ we have $t(x) = \alpha t(1) = 2\alpha$, $n(x,1) = \alpha n(1,1) = 2\alpha n(1) = 2\alpha$ by (1.2), so (1.6) holds for all elements in a degree 2 algebra. ∎

1.7 Remark. If we do not assume t, n are linear, quadratic then A_Ω need not remain of degree 2. For example, if Φ is the field $\mathbb{Z}_2$ then any Boolean (associative) ring is an algebra over $\mathbb{Z}_2$ satisfying $x^2 = x$ for all x, i.e.

$$x^2 - t(x)x + n(x)1 = 0 \quad \text{for } t(x) \equiv 1,\ n(x) \equiv 0 .$$

Clearly t is not linear in this case, and if Ω is a proper extension field of Φ (containing $\omega \neq 1,0$) and A has dimension ≥ 3 then A_Ω is no longer of degree 2: if $1,a,b \in A$ are independent then: $x = a + \omega b \in A_\Omega$ is not of degree 2 (x^2 is not linearly dependent upon 1 and x) because $x^2 = a^2 + \omega(ab+ba) + \omega^2 b^2 = a + \omega^2 b = x + (\omega^2-\omega)b$ and $(\omega^2-\omega)b$ is independent of 1, x(i.e. of 1, a). ∎

By Problem Set 1.3, this is the only alternative example where t is not linear. Notice in this example Φ has only two elements, and A has zero divisors. If Φ were any bigger, or A were free of zero divisors, then A would have been strictly degree 2 because of

1.8 (Degree 2 Criteria) Let A be a unital algebra over a field Φ such that every element $x \in A$ satisfies an equation of degree 2,

$$x^2 + \alpha x + \beta 1 = 0 \qquad (\alpha, \beta \in \Phi, \text{ depending on } x) .$$

Then A will be of degree 2 if either of

(i) Φ contains more than 2 elements

(ii) A is flexible and contains no zero divisors.

In this case we define t and n by $t(x) = -\alpha$, $n(x) = \beta$ if $x \notin \Phi 1$, and $t(x) = 2\lambda$, $n(x) = \lambda^2$ for $x = \lambda 1 \in \Phi 1$.

Proof. $t(x)$ and $n(x)$ are well-defined by the above formula, since if $x \notin \Phi 1$ then the α, β satisfying $x^2 + \alpha x + \beta 1 = 0$ are uniquely determined by x, and it is just a question of whether t is linear (if it is, $n(x)1 = t(x)x - x^2$ is automatically quadratic).

Certainly $t(\lambda x) = \lambda t(x)$: this is trivial if $\lambda = 0$ or $x \in \Phi 1$, while otherwise $\lambda^2\{x^2 - t(x)x + n(x)1\} = 0 = (\lambda x)^2 - t(\lambda x)\lambda x + n(\lambda x)1$ implies $\lambda\{\lambda t(x) - t(\lambda x)\}x = \{\lambda^2 n(x) - n(\lambda x)\}1$; since x, 1 are independent if $x \notin \Phi 1$ both coefficients must vanish, and since $\lambda \neq 0$ we see $t(\lambda x) = \lambda t(x)$, $n(\lambda x) = \lambda^2 n(x)$. It remains only to show $t(x+y) = t(x) + t(y)$.

First consider the case where 1,x,y are dependent; i.e. they lie in a 2-dimensional subspace $\Phi 1 + \Phi z$. It will suffice if t is linear on this subspace, $t(\alpha 1+\beta z) = \alpha t(1) + \beta t(z)$ for all α,β; clearly we need only consider $\alpha,\beta \neq 0$, and dividing by α (recall $t(\lambda x) = \lambda t(x)$) reduces the problem to showing $t(1+w) = t(1) + t(w)$ for any w. Here we may assume $w \notin \Phi 1$, so $t(w+1)(w+1) - n(w+1)1 = (w+1)^2 = w^2 + 2w + 1 =$ $t(w)w - n(w)1 + 2w + 1$ implies (equating coefficients of w) $t(w+1) = t(w) + 2 = t(w) + t(1)$.

Now assume 1,x,y are independent. Then $(x+\lambda y)^2 - x^2 - (\lambda y)^2$ $= \lambda(xy+yx) = \lambda\{(x+y)^2-x^2-y^2\}$ and the formulas for squares give $\{t(x+\lambda y)(x+\lambda y)-n(x+\lambda y)1\} - \{t(x)x-n(x)1\} - \{t(\lambda y)\lambda y-n(\lambda y)1\} =$ $\lambda\{t(x+y)(x+y)-n(x+y)1\} - \lambda\{t(x)x-n(x)1\} - \lambda\{t(y)y-n)1\}$. By independence we can equate coefficients of x and of y to get

$$t(x+\lambda y) - t(x) = \lambda t(x+y) - \lambda t(x) ,$$
$$\lambda t(x+\lambda y) - \lambda^2 t(y) = \lambda t(x+y) - \lambda t(y) .$$

So far we haven't used (i) or (ii). If we assume $|\Phi| > 2$ we can choose $\lambda \neq 0, 1$ in Φ; dividing the second relation by λ and subtracting from the first gives

$$(\lambda-1)\{t(x+y) - t(x) - t(y)\} = 0$$

and we can cancel $\lambda-1 \neq 0$ to get $t(x+y) = t(x) + t(y)$. Thus linearity is easy in case (i).

Suppose now $\Phi = \mathbb{Z}_2$ but that A is flexible without zero divisors. For 1,x,y independent as before we can rewrite $x \circ y = (x+y)^2 - x^2 - y^2$ as

$$(*)\quad x \circ y = \{t(x+y)-t(x)\}x + \{t(x+y)-t(y)\}y - \{n(x+y)-n(x)-n(y)\}1$$

by the formula for squares (compare with (1.1')).

First assume $[x,y] = 0$. Since $\Phi = \mathbb{Z}_2$ we are in characteristic 2, so $[x,y] = 0$ is equivalent to $x \circ y = 0$. Because 1,x,y are independent the coefficients in (*) must all vanish,

$$t(x+y) = t(x) = t(y),\ n(x+y) = n(x) + n(y).$$

Now the assumption that A has no zero divisors guarantees $n(z) \neq 0$ for $z \notin \Phi 1$: if $n(z) = 0$ then $0 = z^2 - t(z)z = z\{z - t(z)1\}$ forces $z = 0$ or $z = t(z)1$. If we also assume $\Phi = \mathbb{Z}_2$ then the only nonzero scalar is 1, so $n(z) = 1$ for $z \notin \Phi 1$. By independence of 1,x,y we know $x,y,x+y \notin \Phi 1$, consequently $n(x+y) = n(x) = n(y) = 1$. But this contradicts $n(x+y) = n(x) + n(y)$, and therefore $[x,y] = 0$ is impossible for 1,x,y independent.

Thus $[x,y] \neq 0$. Commuting the relation (*) with x gives $[x,x \circ y] = \{t(x+y)-t(y)\}[x,y]$. On the other hand, when A is <u>flexible</u> $[x,x \circ y] = -[y,x^2]$ (linearizing $[x,x^2] = 0$, or using flexibility directly $x(xy+yx) - (xy+yx)x = x(xy) - (yx)x = x^2y - [x,x,y] - [y,x,x] - yx^2 = [x^2,y]$) and $-[y,x^2] = -[y,t(x)x - n(x)1] = -t(x)[y,x] = t(x)[x,y]$. Identifying coefficients of $[x,y] \neq 0$ gives $t(x+y) - t(y) = t(x)$. Thus linearity holds in case (ii). ∎

Involutions

We can introduce a **standard involution**

$$x^* = t(x)1 - x \tag{1.9}$$

in any degree 2 algebra. The map $x \to x^*$ is clearly linear of period 2, $x^{**} = \{t(x)1 - x\}^* = t(x)1^* - x^* = t(x)1 - x^* = x$ since $1^* = t(1)1 - 1 = 2 - 1 = 1$ by (1.2). Thus * is a linear involution.

The degree 2 equation (1.1) becomes $n(x)1 = t(x)x - x^2 = x\{t(x)1 - x\} = xx^*$, so

$$x + x^* = t(x)1 \tag{1.10}$$

(Trace and Norm Formulas)

$$xx^* = x^*x = n(x)1 \ . \tag{1.11}$$

Thus the standard involution has the property that all norms xx^* and traces $x + x^*$ lie in $\Phi 1$. In general, an involution * on a linear algebra A is called a **scalar involution** if all norms and traces are scalars in $\Phi 1$: $xx^* = n(x)1$ and $x + x^* = t(x)1$.

Despite its name, the standard involution need not be an involution of algebras. To measure how far it deviates from the involution condition $(xy)^* = y^*x^*$ we write $y = z^*$ and compute $(xz^*)^* - zx^* = t(xz^*)1 - xz^* - zx^* = t(xz^*)1 - n(x,z)1$. The condition that * be an involution is therefore

(1.12) $n(x,y) = t(xy^*) = t(x)t(y) - t(xy)$ (Involution Condition)

Notice that since $n(x,y)$ is symmetric, (1.12) implies

(1.13) $$t(xy) = t(yx) .$$

For a flexible algebra the involution condition is always met.

(1.14) (Scalar Involution Criterion) If A is a flexible degree 2 algebra the standard involution $x^* = t(x)1 - x$ is a scalar involution on A. Conversely, if an algebra A has a scalar involution * it is of degree 2 over Φ and the standard involution associated with A is just *.

Proof. Assume A is flexible of degree 2. We know

$0 = x(x \circ y) - x \circ xy$ $\quad (L_x(L_x+R_x) = (L_x+R_x)L_x$ by flexibility)

$= x\{t(x)y+t(y)x-n(x,y)1\} - \{t(x)xy+t(xy)x-n(x,xy)1\}$ (by (1.1'))

$= t(y)\{t(x)x-n(x)1\} - n(x,y)x - t(xy)x + n(x,xy)1$ (by (1.1) on x)

$= \{t(x)t(y) - t(xy) - n(x,y)\}x + \{n(x,xy)-t(y)n(x)\}1.$

If $x \notin \Phi 1$ then by independence the coefficients of x and 1 must vanish, hence $t(x)t(y) - t(xy) - n(x,y) = 0$ as in (1.12). On the other hand, (1.12) is trivial for $x = \alpha 1 \in \Phi 1$ since $t(1) = 2$ by (1.2) and $n(1,y) = t(y)$ by (1.6). Thus the Involution Condition is met for all x,y when A is flexible.

If A has a scalar involution we can run the trace and norm formulas $x + x^* = t(x)1$, $xx^* = n(x)1$ backwards to get the degree 2 equation: $n(x)1 = xx^* = x\{t(x)1 - x\} = t(x)x - x^2$ becomes

$$x^2 - t(x)x + n(x)1 = 0$$

where t, n are automatically linear and quadratic. This gives (1.1), and (1.2) comes from $t(1)1 = 1 + 1^* = 2$, $n(1)1 = 1 \cdot 1^* = 1$. Therefore A is degree 2. Its standard involution $x^* = t(x)1 - x$ coincides with the original involution. ∎

In particular, all Cayley algebras are of degree 2, which is the reason we are interested in degree 2 algebras.

In the first part of the above proof the coefficient $n(x,xy) - t(y)n(x)$ of 1 vanished for $x \notin \Phi 1$; on the other hand

$$n(x,xy) = t(y)n(x)$$

is trivial for $x = \alpha 1 \in \Phi 1$ since $n(1) = 1$ by (1.2) and $n(1,y) = t(y)$ by (1.6) again. Therefore the relation holds for all x. Linearization gives $n(x,zy) + n(z,xy) = t(y)n(x,z)$ or $n(xy,z) = n(x,t(y)z) - n(x,zy) = n(x,zy^*)$.

Thus right multiplication R_y by y becomes right multiplication by y* when moved across the bilinear norm form, or in other words the adjoint relative to the bilinear form is $R_y^* = R_{y^*}$. Dually $L_y^* = L_{y^*}$. Changing notation we have

$$n(xa,b) = n(a,x^*b)$$

(1.15) (Adjoint formulas)

$$n(ax,b) = n(a,bx^*)$$

in any <u>flexible</u> degree 2 algebra. These formulas are very useful for shifting factors from one side to another.

When the algebra is <u>alternative</u> of degree 2 the norm has the additional property that it **permits composition** in the sense that it is multiplicative: the norm of a product is the product of the norms,

(1.16) $$n(xy) = n(x)n(y) .$$

This most useful property will be used to characterize degree 2 algebras in the next section; it can be derived directly from alternativity and the involution: $n(xy) = (xy)(xy)^* = (xy)(y^*x^*)$ $= (xy)(y^*\{t(x)-x\}) = t(x)(xy)y^* - (xy)(y^*x) = t(x)x(yy^*) - x(yy^*)x$ (middle Moufang) $= \{t(x)x-x^2\}n(y) = n(x)n(y)$.

The above proof shows how useful the involution can be in proving results about degree 2 algebras. The involution also leads to succinct formulations for inverses, U-operators, and isotopes.

1.17 (Inverse Criterion) An element x of an alternative algebra of degree 2 is invertible iff its norm is nonzero, in which case

$$x^{-1} = n(x)^{-1} x^* .$$

Proof. The relation $xx^* = x^*x = n(x)1$ shows $y = n(x)^{-1} x^*$ is the inverse of x ($xy=yx=1$) when $n(x) \neq 0$, and when $n(x) = 0$ the relation $xx^* = 0$ shows x is a zero divisor (note $x^*\neq 0$ if $x\neq 0$) and therefore not invertible. (Alternately, if x is invertible then $1=n(1)=n(xy)=n(x)n(y)$ shows $n(x)\neq 0$.) ∎

1.18 Corollary. An alternative algebra of degree 2 is a division algebra iff its norm form is anisotropic (does not represent zero);

$$n(x) = 0 \text{ only for } x = 0 . \blacksquare$$

In the case of an alternative degree 2 algebra, the U-operator also takes on a particularly simple form in terms of the standard involution:

(1.19a) $$U_x y = n(x,y^*)x - n(x)y^*$$

(1.19b) $$U_x y^* = n(x,y)x - n(x)y \qquad \text{(U-formulas)}$$

(1.19c) $$U_x x^* = n(x)x .$$

It suffices to prove (b) (for (c) recall $n(x,x) = 2n(x)$): $U_x y^* = xy^*x = (xy^* + yx^*)x - y(x^*x)$ (by alternativity $[y,x^*,x] = [y,t(x)1 - x,x] = - [y,x,x] = 0$) $= n(x,y)x - n(x)y$ (using (1.11) and its linearization).

Any isotope of a degree 2 alternative algebra remains degree 2, indeed

1.20 (Isotope Formula) If A is a degree 2 alternative algebra then any isotope $A^{(u,v)}$ is again a degree 2 algebra,

$$x^{2(u,v)} - t^{(u,v)}(x)x + n^{(u,v)}(x)1^{(u,v)} = 0$$

where $t^{(u,v)}(x) = n(x,(uv)^*)$ and $n^{(u,v)}(x) = n(uv)n(x)$.

Proof. In $A^{(u,v)}$ the square is $x^{2(u,v)} = U_x(uv) = n(x,(uv)^*)x - n(x)(uv)^*$ (by the U-formula 1.19) where $(uv)^* = n(uv)(uv)^{-1}$ (by 1.17) $= n(uv)1^{(u,v)}$, so $x^{2(u,v)} = n(x,(uv)^*)x - n(x)n(uv)1^{(u,v)} = t^{(u,v)}(x)x - n^{(u,v)}(x)1^{(u,v)}$. Clearly the new $t^{(u,v)}$, $n^{(u,v)}$ are linear and quadratic if the old ones were. ∎

Example: Construction of Degree 2 Algebras

We now construct all degree 2 algebras A over a field Φ of characteristic $\neq 2$. Since $t(1) = 2 \neq 0$ by assumption on the characteristic, any such algebra has a vector space decomposition

$$(1.21) \qquad A = \Phi 1 \oplus A_o \qquad (A_o = \operatorname{Ker} t = \{x \in A \mid t(x) = 0\}).$$

We define a bilinear form σ and an alternating bilinear product $\times$ on the space A_o by

$$(1.22) \qquad \begin{aligned} \sigma(a,b) &= \tfrac{1}{2}t(ab) \\ a\times b &= ab - \sigma(a,b)1. \end{aligned} \qquad (a,b \in A_o)$$

Clearly both of these are bilinear. Note that $a\times b$ is back in A_o because $t(a\times b) = t(ab) - 2\sigma(a,b) = 0$. To see the product is alternating, observe first

$$(1.23) \qquad \sigma(a,a) = -n(a)$$

since taking traces of $a^2 = t(a)a - n(a)1 = -n(a)1$ for $a \in A_o$ yields $t(a^2) = -2n(a)$, or $\sigma(a,a) = -n(a)$. Thus $a\times a = a^2 - \sigma(a,a)1 = a^2 + n(a)1 = a^2 - t(a)a + n(a)1 = 0$ for $a \in A_o$.

Conversely, if we are given a bilinear form σ and an alternating product $\times$ on a vector space A_o we can form an algebra

$$A(A_o, \sigma, \times)$$

by taking A to be $\Phi 1 \oplus A_o$ as a vector space, where the product of $x = \alpha 1 + a$ and $y = \beta 1 + b$ ($\alpha, \beta \in \Phi$, $a, b \in A_o$) is defined as

$$(1.24) \qquad xy = \{\alpha\beta + \sigma(a,b)\}1 \oplus \{\alpha b + \beta a + a\times b\}.$$

Clearly A is a linear algebra with unit 1, and it is degree 2 because

$$\begin{aligned} x^2 &= (\alpha 1 + a)^2 \\ &= \{\alpha^2 + \sigma(a,a)\}1 \oplus \{2\alpha a + a\times a\} \\ &= \{\alpha^2 + \sigma(a,a)\}1 \oplus 2\alpha a \qquad\qquad (\times \text{ is alternating}) \\ &= 2\alpha\{\alpha 1 \oplus a\} - \{\alpha^2 - \sigma(a,a)\}1 \\ &= t(x)x - n(x)1 \end{aligned}$$

for

$$\text{(1.25)} \qquad \begin{aligned} t(x) &= t(\alpha 1 + a) = 2\alpha \\ n(x) &= n(\alpha 1 + a) = \alpha^2 - \sigma(a,a). \end{aligned}$$

These constructions are inverses. If we start from a degree 2 algebra A, construct σ and $\times$ on A_o, then construct $A(A_o,\sigma,\times)$ we get A back again: $A = \Phi 1 \oplus A_o$ is canonically isomorphic to $A(A_o,\sigma,\times)$ as a vector space, and the products in both algebras are the same since 1 acts as unit and the product A_o is given by

$$ab = \sigma(a,b)1 + a\times b \qquad (a,b \in A_o)$$

according to (1.22) for A and (1.24) for $A(A_o,\sigma,\times)$. Note also that the trace and norm given by (1.25) are the original trace and norm: for $x = \alpha 1 + a \in A$ we have $t(x) = 2\alpha$ since $t(a) = 0$ by definition of $a \in A_o$, hence $n(x) = \alpha^2 n(1) + \alpha n(1,a) + n(a)$ $= \alpha^2 + \alpha t(a) + n(a)$ (by (1.2) and (1.6)) $= \alpha^2 + n(a)$ $= \alpha^2 - \sigma(a,a)$ (by (1.23)).

On the other hand, if we start with σ and $\times$ on A_o, construct $A = A(A_o,\sigma,\times)$ of degree 2, then the form σ' and product $\times'$

constructed from A on Ker $t = A_o$ (by (1.25)) are $\sigma'(a,b)$ $= \frac{1}{2}t(ab)$ (by (1.22)) $= \frac{1}{2} \cdot 2\{\sigma(a,b)\}$ (by (1.25) and (1.24)) $= \sigma(a,b)$ and $a\times' b = ab - \sigma'(a,b)1$ (by(1.22)) $= ab - \sigma(a,b)1$ $= a\times b$ (by (1.24)).

Thus there is a 1-1 correspondence between triples $(A_o,\sigma,\times)$ and degree 2 algebras $A = A(A_o,\sigma,\times)$. This establishes the first part of

1.26 (Degree 2 Construction Theorem) The degree 2 algebras over a field Φ of characteristic $\neq 2$ are precisely all

$$A = A(A_o,\sigma,\times)$$

where σ is a bilinear form and $\times$ an alternating bilinear product on A_o.

The standard involution * on $A = A(A_o,\sigma,\times)$ is a true involution of algebras iff

(i) σ is symmetric.

A is flexible iff σ is symmetric and

(ii) $\sigma(a\times b,a) = 0$

A is alternative iff σ is symmetric, satisfies (ii), and also satisfies

(iii) $a\times(a\times b) = \sigma(a,a)b - \sigma(a,b)a.$

Proof. The Involution Condition (1.11) shows * is an involution iff $n(x,y) - t(x)t(y) + t(xy) = \{2\alpha\beta - \sigma(a,b) - \sigma(b,a)\}$ $- (2\alpha)(2\beta) + 2\{\alpha\beta + \sigma(a,b)\}$ (by (1.25) and (1.24)) $= \sigma(a,b) - \sigma(b,a)$ vanishes identically, i.e. iff σ is symmetric.

In (1.12) we saw a necessary condition for * to be an involution was $t(xy) = t(yx)$. Since $t(xy) = t(yx) \Leftrightarrow \sigma(a,b) = \sigma(b,a)$ by (1.25) and (1.24), we now see it is necessary and sufficient in characteristic $\neq 2$.

Flexibility means $[x,y,x] = [\alpha 1 + a, \beta 1 + b, \alpha 1 + a] = [a,b,a]$ vanishes identically, or L_a commutes with R_a on $b \in A_o$, equivalently L_a commutes with $L_a + R_a$ on b: $a(b \circ a) = (ab) \circ a$. Here $a(b \circ a) - (ab) \circ a = a\{t(a)b + t(b)a - n(a,b)1\} - \{t(ab)a + t(a)ab - n(ab,a)1\}$ (by 1.1')) $= -n(a,b)a - t(ab)a + n(ab,a)1$ ($t(a) = t(b) = 0$) $= \{\sigma(a,b) + \sigma(b,a)\}a - 2\sigma(a,b)a - \sigma(a \times b, a)1$ (by (1.23), (1.24), (1.25)). Identifying coefficients in A_o and $\Phi 1$, the conditions for flexibility are

$$\text{(i)} \quad \sigma(a,b) - \sigma(b,a) = 0$$

$$\text{(ii)} \quad \sigma(a \times b, a) = 0$$

Note that this gives an alternate proof of (1.13) that flexibility implies * is an involution.

Left alternativity means the vanishing of $[x,x,y] = [\alpha 1 + a, \alpha 1 + a, \beta 1 + b] = [a,a,b] = a^2 b - a(ab) = \sigma(a,a)b - a\{\sigma(a,b)1 + a \times b\} = \sigma(a,a)b - \sigma(a,b)a - \{\sigma(a, a \times b)1 + a \times (a \times b)\}$. Identifying components in $\Phi 1$ and A_o, the conditions for left alternativity are

$$\text{(ii)}' \quad \sigma(a, a \times b) = 0$$

$$\text{(iii)} \quad a \times (a \times b) = \sigma(a,a)b - \sigma(a,b)a$$

Thus alternativity (= left alternativity + flexibility) is equivalent to (i) + (ii) + (iii). ∎

1.27 Remark. From this we can easily construct examples of degree 2 algebras A where

(i) * is not an involution

(take any non-symmetric σ),

(ii) * is an in volution but A is not flexible

(take σ symmetric but choose $\times$ so $\sigma(a\times b,a) \not\equiv 0$),

(iii) A is flexible but not alternative

(for example, choose $\sigma \equiv 0$ but $a\times(a\times b) \not\equiv 0$).

In Problem Set VII.1.2 we will see how to construct all degree 2 algebras in characteristic 2.

VII. 1 Exercises

1.1 Show that any (unital) subalgebra and any homomorphic image of a degree 2 algebra is degree 2.

1.2 Show that any 2 elements in an alternative degree 2 algebra generate (together with the unit) a subalgebra spanned by 4 elements. Write down a multiplication table. Conclude that any 2 elements generate an associative subalgebra.

1.3 Show that any unital algebra of dimension 2 over a field Φ is automatically of degree 2.

1.4 Multiply (1.1') on both sides to show simultaneously that in an alternative degree 2 algebra $n(xy) = n(x)n(y)$ and $n(x,y) = t(xy^*)$.

1.5 Show that the standard involution in a degree 2 algebra is a true involution iff $[x,y,x] \in \Phi 1$ for all x,y, and deduce anew that the Involution Criterion is met by all flexible algebras.

1.6 Show that a degree 2 algebra is flexible iff $n(x,y) = t(x)t(y) - t(xy)$ and $n(x,xy) = t(y)n(x)$ for all x,y. Give an example of a degree 2 algebra which is not flexible yet * is a true involution.

1.7 In $A = A(A_o, \sigma, \times)$ of degree 2 derive a formula for $n(xy) - n(x)n(y)$ and use it to show n permits composition iff

(i) $\sigma(a,b) = \sigma(b,a)$

(ii) $\sigma(a\times b, a) = 0$

(iii) $\sigma(a\times b, a\times b) = \sigma(a,b)^2 - \sigma(a,a)\sigma(b,b)$.

Deduce that if n permits composition then A is flexible If A is alternative (so(i),(ii) hold), apply σ to $a\times(a\times b) = \sigma(a,a)b - \sigma(a,b)a$ and use linearized (ii) to derive (iii) directly.

VII.1.1 Problem Set on Arbitrary Φ

1. Define degree 2 algebra A and standard involutions * over an arbitrary ring of scalars Φ. Show that all scalar extensions A_Ω remain degree 2.
2. Show (1.1) need not imply (1.2) whenever Φ does not act faithfully on A.
3. If t,n satisfy (1.1) show $t(1)1 = 2$ iff $n(1)1 = 1$. If Φ acts faithfully, conclude $t(1) = 2$ iff $n(1) = 1$.
4. If A contains "independent" elements 1, x such that $\alpha 1 + \beta x = 0$ implies $\alpha = \beta = 0$, show (1.1) implies (1.2).
5. Do Ex. 1.2 over general Φ.
6. If A has a scalar involution, show A is of degree 2 over $\Phi 1$. (WARNING: Since we are not assuming Φ acts faithfully on A, cannot identify Φ with $\Phi 1$; in general, $n(x), t(x) \in \Phi 1$ can't be lifted to quadratic and linear $\tilde{n}(x), \tilde{t}(x) \in \Phi$ satisfying $\tilde{n}(x)1 = n(x), \tilde{t}(x)1 = t(x)$.)
7. Find the Involution Condition, and show that it reduces to (1.12) when Φ acts faithfully.
8. Show that if A is a flexible degree 2 algebra and Φ contains no nilpotent elements then the standard involution is a true involution.
9. Show that an element of an alternative algebra with scalar involution is invertible iff its norm is invertible in Φ.

Deduce that if A is an alternative division algebra of degree 2 over Φ, where Φ acts faithfully on A, then Φ must be a field. Give an example where Φ is unfaithful and an un-field.

The next series of exercises lead to a construction of an associative degree 2 algebra which does not permit composition and on which * is not an involution. Throughout we let M be a Φ-module and $\sigma: M\times M \to \Phi$ a bilinear form on M.

10. If we take $A = \Phi 1 \oplus M$ with product determined by $1\cdot x = x\cdot 1 = x$ and $m_1 m_2 = \sigma(m_1,m_2)1$, show A is a degree 2 algebra over Φ with $t(x) = 2\alpha$, $n(x) = \alpha^2 - \sigma(m,m)$ for $x = \alpha 1 + m$.

11. If σ is chosen so that its values annihilate M, $\sigma(M,M)M = 0$ (if Φ were a field, or more generally M were torsion-free, this would force $\sigma = 0$) show A is associative. If $x_i = \alpha_i 1 + m_i$ show $t(x_1 x_2^*) - n(x_1,x_2) = \sigma(m_2,m_1) - \sigma(m_1,m_2)$; deduce that * is an involution iff σ is symmetric. Show $n(x_1 x_2) - n(x_1)n(x_2) = \alpha_1\alpha_2\{\sigma(m_1,m_2) - \sigma(m_2,m_1)\} + \{\sigma(m_1,m_2)^2 - \sigma(m_1,m_1)\sigma(m_2,m_2)\}$; deduce that n permits composition iff σ is symmetric and $\sigma(m_1,m_2)^2 = \sigma(m_1,m_1)\sigma(m_2,m_2)$ for all $m_1, m_2 \in M$. (The latter condition holds automatically: $\sigma(M,M)M = 0$ implies $\sigma(M,M)\sigma(M,M) = 0$. In particular, the elements of $\sigma(M,M)$ are nilpotent).

12. Conclude that if σ is a non-symmetric bilinear form with $\sigma(M,M)M = 0$ then $A = A(M,\sigma)$ is an associative degree 2 algebra which neither admits composition nor has * an involution. As an

example take $\Phi = \Phi_o[\varepsilon]$ where $\varepsilon^2 = 0$, $M = \Phi_o m_1 \oplus \Phi_o m_2$ free over Φ_o with $\varepsilon M = 0$, σ defined by $\sigma(m_i, m_i) = 0$, $\sigma(m_1, m_2) = \varepsilon$, $\sigma(m_2, m_1) = \alpha_o \varepsilon \neq \varepsilon$.

VII.1.2 Problem Set: Construction in characteristic 2

We will show how to construct all degree 2 algebras A over a field Φ of characteristic 2. We first consider the <u>traceless</u> case where $t(x) = 0$ for all x.

1. If A is traceless of degree 2 then $x^2 = n(x)1$ for all x. Let $A = \Phi 1 \oplus A_o$ be <u>any</u> vector space decomposition of A, with projection π_1 on $\Phi 1$ and π_o on A_o. Define a bilinear form σ and alternating product $\times$ on A_o by

$$ab = \sigma(a,b)1 \oplus a\times b$$

(i.e. $a\times b = \pi_o(ab)$ and $\sigma(a,b)1 = \pi_1(ab)$). Show $\times$ is alternating and $\sigma(a,a) = n(a)$.

2. Given a bilinear form σ and alternating product $\times$ on A_o show the algebra $A(A_o,\sigma,\times) = \Phi 1 \oplus A_o$ with product

$$xy = (\alpha 1 + a)(\beta 1 + b) = \{\alpha\beta + \sigma(a,b)\}1 \oplus \{\alpha b + \beta a + a\times b\}$$

defines a traceless degree 2 algebra with norm $n(x) = n(\alpha 1 + a) = \alpha^2 - \sigma(a,a)$. Conclude all traceless degree 2 algebras in characteristic 2 have the form $A \cong a(A_o,\sigma,\times)$ for some $A_o,\sigma,\times$.

Thus the construction which in characteristic $\neq 2$ gives all degree 2 algebras gives, in characteristic 2, precisely the traceless degree 2 algebras. Moreover, the triple $(A_o,\sigma,\times)$ is not uniquely determined by A, and the conditions under which two such triples give isomorphic algebras seem to be messy.

3. In the traceless case the standard involution degenerates, $x^* = t(x)1 - x = x$. Show $A = A(A_o,\sigma,\times)$ is commutative iff σ is symmetric iff $n(x,y) \equiv 0$; show A is flexible iff $n(x,y) \equiv 0$ iff A is commutative iff (i) $\sigma(a,b) = \sigma(b,a)$; show A is left alternative iff (ii) $\sigma(a,a\times b) \equiv 0$, (iii) $a\times(a\times b) = \sigma(a,a)b - \sigma(a,b)a$; show A is alternative iff (i), (ii), (iii) (in which case A is commutative and associative, thus merely a scalar extension A of Φ of "exponent" 1).

Agreeing that traceless degree 2 alternative algebras are unrewarding, we pass to the traceful case $t \neq 0$.

4. In such a traceful degree 2 algebra A we fix a vector u with $t(u) = 1$. Show we have a vector-space decomposition $A = \Phi 1 \oplus \Phi u \oplus A_o$ for $A_o = (\Phi 1 + \Phi u)^{\perp} = \{x\in A \mid n(1,x) = n(u,x) = 0\}$ since $\Phi 1 + \Phi u$ is a hyperbolic plane relative to the bilinear form $n(\cdot,\cdot)$ (in the sense that $n(1,1) = n(u,u) = 0$, $n(1,u) = n(u,1) = 1$). Show the product of $a,b\in A_o$ is given by

$$(1)\qquad ab = \sigma(a,b)1 \oplus \tau(a,b)u \oplus a\times b$$

where σ,τ are bilinear forms and $\times$ a bilinear product satisfying

$$\sigma(a,b) = n(ab,u)$$

$$(1')\qquad \tau(a,b) = n(ab,1) = t(ab)$$

$$\sigma(a,a) = n(a) \qquad \tau(a,a) = 0 \qquad a\times a = 0.$$

In particular, τ and $\times$ are alternating. Show the product of u with $a\in A_o$ is

$$(2)\qquad au = \sigma(a)1 \oplus \tau(a)u \oplus a\times u$$

$$ua = \sigma(a)1 \oplus \tau(a)u \oplus u\times a$$

where σ,τ are linear functionals and $u\times, \times u$ linear transformations on A_o given by

$$(2') \qquad \sigma(a) = n(ua,u),\ \tau(a) = n(ua,1) = t(ua),\ a\times u + u\times a = a.$$

Show $u^2 = u + \mu 1$ for $\mu = n(u)$. Deduce that the product of $x = \alpha 1 + \gamma u + a$, $y = \beta 1 + \delta u + b$ is given by the egregious formula

$$(3) \qquad \begin{aligned} xy = {} & \{\alpha\beta + \gamma\delta\mu + \sigma(a,b) + \gamma\sigma(b) + \delta\sigma(a)\}1 \oplus \\ & \{\alpha\delta + \delta\gamma + \gamma\beta + \tau(a,b) + \gamma\tau(b) + \delta\tau(a)\}u \oplus \\ & \{\alpha b + \beta a + a\times b + \gamma u\times b + \delta a\times u\} \end{aligned}$$

Show the trace and norm of $x = \alpha 1 + \gamma u + a$ are

$$(4) \qquad \begin{aligned} t(x) &= \gamma \\ n(x) &= \alpha^2 + \alpha\gamma + \gamma^2\mu + \sigma(a,a). \end{aligned}$$

5. Conversely, given bilinear forms σ,τ on A_o with τ alternating, linear forms σ,τ on A_o, a bilinear alternating product $\times$ on A_o, a scalar $\mu\varepsilon\Phi$, and linear transformations $u\times$ and $\times u$ on A_o with $u\times a + a\times u = a$, show that the algebra

$$A = A(A_o,\sigma,\tau,\sigma,\tau,\mu,\times,u\times,\times u)$$

which is $A = \Phi 1 \oplus \Phi u \oplus A_o$ as vector space, together with the product (3), is a degree 2 algebra with norm and trace (4).

6. Show the constructions $A \leftrightarrow (A_o,\sigma,\tau,\sigma,\tau,\mu,\times,u\times,\times u)$ are inverses, so that the traceful degree 2 algebras in characteristic 2 are precisely the $A(A_o,\sigma,\tau,\sigma,\tau,\mu,\times,u\times,\times u)$.

7. Show $A = A(A_o,\sigma,\tau,\sigma,\tau,\mu,\times,u\times,\times u)$ is commutative iff $A_o = 0$; show $*$ is an involution iff (i) $\tau(a,b) = \sigma(a,b) + \sigma(b,a)$ and (ii) $\tau(a) = 0$ for all $a,b\varepsilon A_o$; show A is flexible iff (i), (ii), and (iii) $\sigma(a) = 0$, (iv) $\sigma(a,b) = \tau(a,u\times b)$, (v) $\tau(a,a\times b) = 0$; show A is alternative iff (i)-(v) hold as well as

(vi) $u \times (u \times a) = u \times a + \mu a$

(vii) $u \times (a \times b) + a \times (u \times b) = a \times b$

(viii) $a \times (a \times b) = \sigma(a,a)b + \sigma(a,b)a + \tau(a,b)a \times u.$

VII. 1.3 Problem Set: Algebras which are almost degree two

In this set of problems we investigate algebras in which every element satisfies a quadratic equation, but the trace is not linear. Throughout A denotes a unital algebra over a field Φ which is **almost degree 2** in the sense that each element $x \in A$ satisfies an equation $x^2 - \alpha x + \beta 1 = 0$ for some $\alpha, \beta \in \Phi$.

1. Define $t(x) = \alpha$, $n(x) = \beta$ if $x \notin \Phi 1$ and $t(\alpha 1) = 2\alpha$, $n(\alpha 1) = \alpha^2$. Show $t(\lambda x) = \lambda t(x)$, $n(\lambda x) = \lambda^2 n(x)$, $t(x+\lambda 1) = t(x) + 2\lambda$, $n(x+\lambda 1) = n(x) + \lambda t(x) + n(\lambda 1)$.
2. To measure how far the trace t is from being linear, define $\Delta(x,y) = t(x+y) - t(x) - t(y)$. Thus t is linear iff $\Delta \equiv 0$. Show (i) $\Delta(x,y) = 0$ if x,y are dependent; (ii) $\Delta(x,\lambda y) = \lambda\Delta(x,y) = \Delta(x,y)$ if $\lambda \neq 0$, and conclude $\Delta \equiv 0$ if $|\Phi| > 2$; (iii) $\Delta(x,1) = 0$; (iv) $\Delta(x,y+z) - \Delta(x+y,z) = \Delta(x,y) - \Delta(y,z)$; (v) $\Delta(x,y+\lambda x+\mu 1) = \Delta(x,y)$; (vi) if $x,y,z,1$ are independent $\Delta(x,y) = \Delta(y,z) = \Delta(z,x)$.
3. If A is not of degree 2 then $\Phi = \mathbb{Z}_2$, so each $\Delta(x,y)$ is 1 or 0 (with at least one pair where Δ takes the value 1). Show that if $\Delta(x_0,y_0) = 1$ then $\Delta(x,y) = 1$ iff $x,y,1$ independent. Show the "anti-trace" $s(x)$ (defined to be 0 for $x \in \Phi 1$ and $1 + t(x)$ for $x \notin \Phi 1$) is additive: $s(x+y) = s(x) + s(y)$. Show $s \equiv 0$ iff $x \circ y = xy + yx \in \Phi 1$ for all x,y.

4. If A is not of degree 2 show $[x,x,y] + [x,y,x] + [y,x,x] = x \circ y$ for all x,y. Show A is commutative iff $[x,x,y] + [x,y,x] + [y,x,x] \equiv 0$ (in which case $s \equiv 0$ also). Give an example where $s \equiv 0$ yet A is not commutative.

5. Prove Slater's Almost-but-not-quite Theorem: If A is alternative and almost degree 2, but is not of degree 2, then A is Boolean: $x^2 = x$ for all $x \in A$.