§5. Connectivity

Idempotents can be connected, strongly connected, or interconnected. Interconnectivity is the more general notion, and is really a property of the Peirce decomposition; any Peirce decomposition of a simple algebra is interconnected. We show an n-interconnected algebra for $n \geq 3$ is strongly associative: it and all its bimodules are associative, and it remains nuclear in all larger algebras. Thus simple alternative algebras with three orthogonal idempotents are associative.

The basic idea here is that if two idempotents belong to the "same piece" of an algebra (such as the same component of a semisimple algebra), they ought to be "connected" with each other, and that if we can tie all idempotents together we will have tied the whole algebra together.

We say two orthogonal idempotents e_1, e_2 are **connected** if there are elements x_{12}, x_{21} in the Peirce spaces A_{12}, A_{21} such that $x_{11} = x_{12}x_{21}$ is invertible in A_{11} and $x_{22} = x_{21}x_{12}$ is invertible in A_{22}. They are **strongly connected** if we can actually choose x_{12}, x_{21} so $x_{11} = e_1$, $x_{22} = e_2$: $x_{12}x_{21} = e_1$ and $x_{21}x_{12} = e_2$. As in the associative case, these two notions coincide.

5.1 (Strong Connection Lemma) Two orthogonal idempotents e_1, e_2 an alternative algebra are connected iff they are strongly connected: if there are $x_{12} \in A_{12}$ and $x_{21} \in A_{21}$ such that $x_{12}x_{21} = x_{11}$ is invertible in A_{11} and $x_{21}x_{12} = x_{22}$ is invertible in A_{22}, then $y_{12}y_{21} = e_1$, $y_{21}y_{12} = e_2$ for $y_{12} = x_{11}^{-1}x_{12} = x_{12}x_{22}^{-1}$, $y_{21} = x_{21}$.

Proof. First observe that the two expressions for y_{12} coincide: by flexibility $x_{11}x_{12} = (x_{12}x_{21})x_{12} = x_{12}(x_{21}x_{12}) = x_{12}x_{22}$, so (multiplying by x_{11}^{-1} and x_{22}^{-1}) $x_{11}^{-1}(x_{11}x_{12})x_{22}^{-1} = \{(x_{11}^{-1}x_{11})x_{12}\}x_{22}^{-1} = \{e_1x_{12}\}x_{22}^{-1} = x_{12}x_{22}^{-1}$ coincides with $x_{11}^{-1}(x_{12}x_{22})x_{22}^{-1} = x_{11}^{-1}\{x_{12}(x_{22}x_{22}^{-1})\} = x_{11}^{-1}\{x_{12}e_2\} = x_{11}^{-1}x_{12}$.

By Peirce associativity (3 Peirce spaces) we get $y_{12}y_{21} = (x_{11}^{-1}x_{12})x_{21} = x_{11}^{-1}(x_{12}x_{21}) = x_{11}^{-1}x_{11} = e_1$ immediately, and similarly $y_{21}y_{12} = x_{21}(x_{12}x_{22}^{-1}) = (x_{21}x_{12})x_{22}^{-1} = x_{22}x_{22}^{-1} = e_2$. ∎

We know in the associative case that in a matrix algebra $M_n(D)$ the diagonal idempotents e_{ii} are strongly connected by the matrix units e_{ij}, $e_{ij}e_{ji} = e_{ii}$. In a split Cayley algebra the same holds (for one thing, a split Cayley contains a split quaternion $\cong M_2(\Phi)$). Examples of idempotents which are <u>not</u> connected are those lying in different components of a direct sum $A_1 \boxplus A_2$.

To connect all the e_i together, it is enough to connect them all to e_1.

5.2 (Transitivity Lemma) If e_1, e_2 are strongly connected by x_{12}, x_{21} and e_2, e_3 are strongly connected by x_{23}, x_{32} then e_1, e_3 are strongly connected by $x_{13} = x_{12}x_{23}$, $x_{31} = x_{32}x_{23}$.

Proof. Clearly $x_{13} \in A_{12}A_{23} \subset A_{13}$ and $x_{31} \in A_{32}A_{21} \subset A_{31}$ by the Peirce relations, and $x_{13}x_{31} = (x_{12}x_{23})x_{31} = x_{12}(x_{23}x_{31})$ $= x_{12}\{x_{23}(x_{32}x_{21})\} = x_{12}\{(x_{23}x_{32})x_{21}\} = x_{12}\{e_2x_{21}\} = x_{12}x_{21} = e_1$ by Peirce associativity and the assumed connections $x_{23}x_{32} = e_2$, $x_{12}x_{21} = e_1$. Similarly $x_{31}x_{13} = e_3$. ∎

5.3 Corollary. If e_1 is strongly connected to each e_i $(i = 2,\cdots,n)$ by means of elements x_{i1}, x_{1i} then any e_i and e_j are strongly connected by $x_{ij} = x_{i1}x_{1j}$, $x_{ji} = x_{j1}x_{1i}$. ∎

A weaker, therefore more common, notion is that of interconnectivity. We say two orthogonal idempotents e_1, e_2 are **interconnected** if $A_{12}A_{21} = A_{11}$, $A_{21}A_{12} = A_{22}$. It is actually enough if $A_{12}A_{21}$ just contains e_1 and $A_{21}A_{12}$ contains e_2, since if $e_i \in A_{ij}A_{ji}$ then $A_{ii} = A_{ii}e_i \subset A_{ii}(A_{ij}A_{ji}) =$ $(A_{ii}A_{ij})A_{ji} \subset A_{ij}A_{ji}$ by Peirce associativity. The difference between connectivity and interconnectivity is that for (strong) connectivity we must have $e_i = x_{ij}x_{ji}$ expressible as a single product, whereas for interconnectivity we need only have $e_i = \Sigma_\alpha x_{ij}^{(\alpha)} x_{ji}^{(\alpha)}$ a sum of such products.

A family $\{e_i\}$ of orthogonal idempotents is **connected** (resp.

strongly connected, interconnected) if each pair e_i, e_j is.
NOTE THAT IF e_1, e_2 ARE CONNECTED OR INTERCONNECTED IN A THEY REMAIN SO IN ANY EXTENSION $\tilde{A} \supset A$, since the elements $x_{ij}^{(\alpha)}$, $x_{ji}^{(\alpha)}$ which do the connecting or interconnecting in A do it just as well in $\tilde{A}$.

Interconnectivity is really a property of the Peirce components, $A_{ij}A_{ji} = A_{ii}$ for $1 \leq i, j \leq j$, rather than of the idempotents themselves. To take into account the "missing" idempotent $e_0 = 1-e$ ($e = \sum_{i=1}^{n} e_i$), we say a Peirce decomposition $A = \oplus_{i,j=0}^{n} A_{ij}$ relative to $e_1, \cdots, e_n$ is **interconnected** if

$$(5.4) \qquad A_{ij}\, A_{ji} = A_{ii} \qquad (0 \leq i \neq j \leq n).$$

(WARNING: Although A remains interconnected in any extension $\tilde{A}$, an interconnected Peirce decomposition of A need not extend to one of $\tilde{A}$ if e_0 does not exist in A or if e_0 exists but $\tilde{A}$ is not a unital extension).

These rules have as consequence

$$(5.4)' \qquad A_{ij}\, A_{jk} = A_{ij} \qquad (i,j,k \neq).$$

Indeed, if i,j,k are distinct either e_i or e_k exists (i or k isn't 0), and if (say) e_i exists then $A_{ik} = e_i A_{ik} = (A_{ij}A_{ji})A_{ik}$ $= A_{ij}(A_{ji}A_{ik})\ A_{ij}A_{jk}$ by Peirce associativity.

One reason interconnectivity is important is

5.5 (Simple Interconnectivity Theorem) Any Peirce decomposition of a simple alternative algebra is interconnected.

Proof. Let $A = \oplus_{i,j=0}^{n} A_{ij}$ be the Peirce decomposition relative to $e_1,\cdots,e_n$ (tacitly assumed to be nonzero, $e_i \neq 0$, and non supplementary, $e_0 \neq 0$).

If $e = e_k$ $(0 \leq k \leq n)$ the Peirce decomposition relative to e has $A_{10}(e) = \Sigma_{j\neq k} A_{kj}$, $A_{01}(e) = \Sigma_{i\neq k} A_{ik}$, $A_{11}(e) = A_{kk}$, $A_{00}(e) = \Sigma_{i,j\neq k} A_{ij}$ by the Collection Formula 2.5. The fact that $A_{01}(e)A_{10}(e) = A_{00}(e)$ for $e \neq 1,0$ in a simple algebra (Connector Corollary 4.3) implies for $i \neq k$ $A_{ii} = e_i A_{00}(e)e_i = e_i\{A_{01}(e)A_{10}(e)\}e_i = \{e_i A_{01}(e)\}\{A_{10}(e)e_i\} = A_{ik}A_{ki}$ even if $i = 0$ and e_i exists only in $\hat{A}$, or if $k = 0$ so $e = e_k$ exists only in $\hat{A}$ (note in this case $A_{01}(e_0)A_{10}(e_0) = A_{10}(e_1+\cdots+e_n)A_{01}(e_1+\cdots+e_n) = A_{11}(e_1+\cdots+e_n) = A_{00}(e_0)$ since $e_1+\cdots+e_n$ does exist inside A). This establishes (5.4). ∎

We say A is **n-interconnected** if it has an interconnected Peirce decomposition of length n: $A = \oplus_{i,j=0}^{n-1} A_{ij}$ (corresponding to n-1 actual idempotents $e_1,\cdots,e_{n-1}$ and one perhaps fictitious idempotent e_0, making n idempotents in all). ONE OF THE BASIC RESULTS ABOUT ALTERNATIVE ALGEBRAS IS THAT WHEN THEY HAVE ENOUGH IDEMPOTENTS THEY ARE AUTOMATICALLY ASSOCIATIVE. Indeed, interconnected algebras are more than just associative, they are **strongly associative** in the sense that they remain nuclear in any larger algebra: A is strongly associative if $A \subset \tilde{A}$ implies $A \subset N(\tilde{A})$.

5.6 (Strong Associativity Theorem) An n-interconnected alternative algebra ($n \geq 3$) is strongly associative.

Proof. The basic reason is simple: an interconnected algebra A stays interconnected in any larger algebra $\tilde{A}$. If $A = \bigoplus_{i,j=0}^{n-1} A_{ij}$ relative to $e_1, \cdots, e_{n-1}$ with (5.4) $A_{ii} = A_{ij}A_{ji} = A_{ik}A_{ki}$ and (5.4)' $A_{ik} = A_{ij}A_{jk}$ ($i,j,k \neq n \geq 3$) then $\tilde{A} = \bigoplus_{i,j=0}^{n-1} \tilde{A}_{ij}$ relative to $e_1, \cdots, e_{n-1}$ with $A_{ij} \subset \tilde{A}_{ij}$ and hence $A_{ii} = A_{ij}A_{ji} \cap A_{ik}A_{ki} \subset \tilde{A}_{ij}\tilde{A}_{ji} \cap \tilde{A}_{ik}\tilde{A}_{ki} \subset N(\tilde{A})$ (by Peirce Nuclearity 3.19) and $A_{ik} = A_{ij}A_{jk} \subset \tilde{A}_{ij}\tilde{A}_{jk} \subset N(\tilde{A})$ (by Peirce Nuclearity 3.18), so $A \subset N(\tilde{A})$. ∎

In particular A is nuclear in itself, $A = N(A)$.

5.7 (Interconnectivity Theorem) If A is n-interconnected for $n \geq 3$ then A is associative. ∎

As an immediate consequence of the previous theorems, A SIMPLE ALGEBRA WITH ENOUGH IDEMPOTENTS IS ASSOCIATIVE.

5.8 Theorem. Any simple alternative algebra with two nonzero orthogonal dempotents e_1, e_2 with $e_1 + e_2 \neq 1$ is associative. ∎

If M is a bimodule for A then the fact that A is nuclear in the split null extension $E = A \oplus M$ means A acts associatively on M.

5.9 (Bimodule Associativity Theorem) All alternative bimodules for a strongly associative algebra are associative. ▪

5.10 Remark. As an example of the warning after (5.4), if the extension $E = A \oplus M$ is not null, so M has a nontrivial multiplication, then although A acts associatively on M the algebra M itself need not be associative. Of course, if A is <u>unital</u> n-interconnected ($n \geq 3$) and M unital as an A-bimodule, then E is also n-interconnected ($1 = \sum_{i=0}^{n-1} e_i$ still has $e_i \in A_{ij}A_{ji} \subset E_{ij}E_{ji}$) and therefore associative. But if A is not unital, or A is unital but M is not, then $e_0 = 1 - \sum_{i=1}^{n-1} e_i$ doesn't exist in A, and E need not be interconnected: $E_{00} = A_{00} \oplus M_{00}$ need not equal $E_{0i}E_{i0} = (A_{0i} \oplus M_{0i})(A_{i0} \oplus M_{i0}) = A_{0i}A_{i0} \oplus \{A_{0i}M_{i0} + M_{0i}A_{i0}\}$ (there need not be any way of recovering M_{00} from M_{i0} and M_{0i}).

The point is that a unital Peirce decompositon which is interconnected will remain interconnected in any unital extension, but a non-unital Peirce decomposition need not. ▪

We say an idempotent e is **strongly associative** (relative to A) if not merely e but the whole Peirce space $e\tilde{A}e$ is nuclear in any larger algebra $\tilde{A}$ containing A: $A \subset \tilde{A}$ implies $e\tilde{A}e \subset N(\tilde{A})$. For example, the unit element of a 3-interconnected algebra is strongly associative, or more generally

5.11 Lemma. If $e = \Sigma e_i$ is a sum of orthogonal idempotents such that each e_i is interconnected in A to at least two other e_j, e_k (but not necessarily to all other e_ℓ), then e is strongly associative relative to A.

Proof. If $A \subset \tilde{A}$ then $e\tilde{A}e = \Sigma e_i \tilde{A} e_j = \Sigma \tilde{A}_{ij}$. To see that $\tilde{A}_{ii} \subset N(\tilde{A})$ is nuclear, observe that if e_i is interconnected to e_j, e_k in A it remains interconnected in $\tilde{A}$, hence by Peirce Nuclearity 3.17 $\tilde{A}_{ii} = \tilde{A}_{ij}\tilde{A}_{ji} = \tilde{A}_{ik}\tilde{A}_{ki} \subset N(\tilde{A})$. To see $\tilde{A}_{ij} \subset N(\tilde{A})$ for $i \neq j$, observe that e_i is interconnected to some $e_k \neq e_j$ so that $\tilde{A}_{ij} = e_i\tilde{A}_{ij} \subset (\tilde{A}_{ik}\tilde{A}_{ki})\tilde{A}_{ij} = \tilde{A}_{ik}(\tilde{A}_{ki}\tilde{A}_{ij}) \subset \tilde{A}_{ik}\tilde{A}_{kj}$ and therefore $\tilde{A}_{ij}$ is nuclear by 3.17 again. (Alternately, all unruly associators involving $\tilde{A}_{ij}$ in 3.7 vanish since by nuclearity of e_i all $\tilde{A}^2_{ik}$ and $\tilde{A}^2_{ki}$ vanish). ∎

The following technical proposition will allow us to lift nuclear derivations D_n in the proof of Malcev's Theorem in Chapter VIII.

5.12 (Proposition) If A is an alternative algebra with unit e over a field Ω such that some scalar extension $A \otimes_\Omega \Gamma$ becomes n-interconnected ($n \geq 3$), then e is strongly associative (relative to A/Φ whenever Ω/Φ is a separable field extension.

Proof. Suppose we have Φ-algebras $A = eAe \subset e\tilde{A}e \subset \tilde{A}$. To show strong associativity $e\tilde{A}e \subset N(\tilde{A})$ of e in $\tilde{A}$ it suffices to show strong associativity $e(\tilde{A} \otimes_\Phi \Sigma)e \subset N(\tilde{A} \otimes_\Phi \Sigma)$ of e in $\tilde{A} \otimes_\Phi \Sigma$ for $\Sigma \supset \Gamma$ an algebraic closure of Γ, since then $e\tilde{A}e \subset \tilde{A} \cap N(\tilde{A} \otimes_\Phi \Sigma) \subset N(\tilde{A})$. To this end we will show that in $\tilde{A} \otimes_\Phi \Sigma$ we can write $e = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}$ as a sum of orthogonal idempotents such that each f_{ij} is interconnected to $n - 1 \geq 2$ idempotents f_{ik} ; by the Lemma 5.13 this will show e is strongly associative in $\tilde{A} \otimes_\Phi \Sigma$.

Now since Ω/Φ is separable we have $A \otimes_\Phi \Sigma \cong \boxplus A_i$ for $A_i \cong A \otimes_\Omega \Sigma \cong (A \otimes_\Omega \Gamma) \otimes_\Gamma \Sigma$ by the Separable Decomposition Theorem 0.00. By hypothesis $A \otimes_\Omega \Gamma$ (hence also each $A_i \cong (A \otimes_\Omega \Gamma) \otimes_\Gamma \Sigma$) is n-interconnected. Thus the unit e_i of A_i is the sum $e_i = \sum_{j=1}^{n} f_{ij}$ of n interconnected idempotents, and the unit e of $A \otimes_\Phi \Sigma = \boxplus A_i$ may be written $e = \Sigma e_i = \Sigma\Sigma f_{ij}$ in $A \otimes_\Phi \Sigma \subset \tilde{A} \otimes_\Phi \Sigma$ as promised. ∎

VI. 5 Exercises

5.1 If (as in 5.1) $x_{12}z_{21} = x_{11}, w_{21}x_{12} = x_{22}$ are invertible for $x_{12} \in A_{12}, z_{21}, w_{21} \in A_{21}$, is it true that $z_{21}x_{11}^{-1} = x_{22}^{-1}w_{21}$? Prove $x_{12}y_{21} = e_1, y_{21}x_{12} = e_2$ for $y_{21} = z_{21}x_{11}^{-1}$ and for $y_{21} = x_{22}^{-1}w_{21}$.

5.2 Give an example of an idempotent e such that $A_{01}A_{10} = A_{00}, A_{10}A_{01} = A_{11}$ but $x_{00}A_{00} = A_{00}x_{00} = 0$ for some $x_{00} \neq 0$.

5.3 If $A = \oplus_{i,j=0}^{n} A_{ij}$ is an interconnected Peirce decomposition, show (I') $A_{ii}A_{ij} = A_{ij}$ and $A_{ji}A_{ii} = A_{ji}$ if $i \neq 0$, (III') $x_{ii}A_{ij} = 0$ or $A_{ji}x_{ii} = 0$ implies $x_{ii} = 0$. What about (II') $A_{00}A_{0i} = A_{0i}, A_{i0}A_{00} = A_{i0}$?

5.4 If orthogonal idempotents e_1, e_2, e_3 are connected, are $e_1 + e_2$ and e_3 connected? Repeat for the interconnected situation. Generalize.

5.5 If an idempotent e is 2-interconnected, $e = e_1 + e_2$ for e_1, e_2 interconnected, show $A_{10}(e)^2 = A_{01}(e)^2 = 0$ and $A_{10} \subset N(A)$.

5.6 If A is strongly associative show $Su(A) \cong A$, $Mu(A) \cong A \boxplus A^{op}$, any bispecialization or birepresentation (λ, ρ) has λ a left and ρ a right specialization which commute with each other.

5.7 If A_i are strongly associative so is their direct sum $\boxplus A_i$. What about the direct product ΠA_i ?

5.8 If $\Omega \supset \Phi$ is free as a Φ-module, show A/Φ is strongly associative iff A_Ω/Ω is strongly associative. As a consequence, a form of a strongly associative algebra is strongly associative. Conclude that if A_Ω is n-interconnected ($n \geq 3$) then A is strongly associative.

5.9 If A/Ω is strongly associative and Ω/Φ is a separable field extension, show A/Φ is strongly associative.

5.10 Show directly that if e is the unit of an n-interconnected alternative algebra A($n \geq 3$) then e is strongly associative (relative to A).

5.11 Prove the Addition Lemma: if $e = e_1 \oplus \cdots \oplus e_m$ is a direct sum of idempotents e_i strongly associative relative to A_i, then e is strongly associative relative to $A = A_1 \boxplus \cdots \boxplus A_m$.

5.13 Prove the Extension Lemma: If $\Omega \supset \Phi$ is free as a Φ-module, and $e \in A$ is strongly associative relative to A_Ω (as Ω-algebra), it is strongly associative relative to A (as Φ-algebra).

5.14 Show that if A is n-interconnected (not necessarily unital) then every <u>scalar</u> extension A_Ω remains n-interconnected.

5.15 We say the Peirce decomposition is **nondegenerately interconnected** if in addition to 5.4 we have the nondegeneracy condition

(5.4)'' $\qquad x_{00}A_{00} = 0$ or $A_{00}x_{00} = 0$ implies $x_{00} = 0$.

Show this condition is automatic if e_0 exists (ie. if A is unital). Show that any Peirce decomposition of a simple algebra is nondegenerately interconnected. But show the nondegeneracy condition $x_{00}A_{00} = 0 \implies x_{00} = 0$ need not be preserved in a split null extension $E = A \oplus M$, $x_{00}E_{00} = 0$ need not imply $x_{00} = 0$: if $x_{00} = a_{00} + m_{00}$ then $(a_{00} \oplus m_{00})(A_{00} \oplus M_{00}) = a_{00}A_{00} \oplus \{a_{00}M_{00} + m_{00}A_{00}\} = 0$ means $a_{00}A_{00} = a_{00}M_{00} = m_{00}A_{00} = 0$, so $a_{00} = 0$, but there is no reason why $m_{00}A_{00} = 0$ should force $m_{00} = 0$.