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Chapter VI

Idempotents

§1. The nature of idempotents

The basic notions concerning idempotents in alternative algebras are the same as those in associative algebras: we can speak of orthogonal, supplementary, minimal, maximal, and division idempotents.

We have all the concept surrounding idempotents in alternative algebras that we do in associative algebras. An element e is **idempotent** if $e^2 = e$. Trivially 0 and 1 are idempotents; an $e \neq 0, 1$ is called a **proper** idempotent. (Sometimes 0 is not considered an idempotent, but this restriction is inconvenient). **Orthogonality** of idempotents is defined by

$$e \perp f \quad \text{if} \quad ef = fe = 0 .$$

A family $\{e_i\}$ of idempotents is **pairwise orthogonal** if each pair is orthogonal, $e_i \perp e_j$ for $i \neq j$, so that $e_i e_k = \delta_{ik} e_i$. A family of idempotents $e_1, \cdots, e_n$ in a unital algebra are **supplementary** if they add up to 1, $\sum_{i=1}^{n} e_i = 1$.

An important example of supplementary orthogonal idempotents are e and $1-e$. If e is an idempotent so is $1-e$ since $(1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$; it is orthogonal to e since $e(1-e) = (1-e)e = e - e^2 = 0$; clearly e and $1-e$ are supplementary.

Thus we can break 1 apart into orthogonal idempotents. We can also put two or more orthogonal idempotents together: if $\{e_1,\cdots,e_n\}$ are pairwise orthogonal idempotents their sum $e = \sum_{i=1}^{n} e_i$ is an idempotent such that $ee_i = e_i = e_i e$ because $ee_j = (\Sigma e_i)e_j = e_j e_j = e_j$ by orthogonality and $e^2 = \Sigma ee_j = \Sigma e_j = e$.

We can partially order the idempotents by writing

$$(1.1) \qquad e \geq f \quad \text{iff} \quad ef = fe = f \ .$$

Notice that $1 \geq e \geq 0$ for all e. Since $e^2 = e$ we have reflexivity $e \geq e$, clearly $e \geq f$ and $f \geq e$ imply $e = f$, and transitivity $e \geq f \geq g \implies e \geq g$ follows from Moufangitivity $eg = e\{fgf\} = \{(ef)g\}f = \{fg\}f = g$ and dually $ge = g$.

Illumination is cast on this ordering of idempotents by observing that a bigger idempotent is obtained by tacking on an orthogonal piece to a smaller idempotent:

$$(1.2) \qquad e \geq f \quad \text{iff} \quad e = f + g \quad \text{for} \quad f \perp g \ .$$

Clearly $f + g \geq f$ since $(f+g)f = f(f+g) = f^2 = f$ if $fg = gf = 0$, and conversely if $e \geq f$ then $e = f + g$ where $g = e - f$ has $g^2 = e^2 - ef - fe + f^2 = e - 2f + f$ (by $e \geq f$) $= e - f = g$ and $fg = fe - f^2 = f - f = 0$ (by $fe = f$) and dually, so $g^2 = g$ and $fg = gf = 0$.

A maximal (or principal) idempotent is one which is maximal relative to this ordering; equivalently, in view of (1.2), one which has no nonzero idempotent orthogonal to it. If it exists, 1 is the unique maximal idempotent. Usually one considers maximal idempotents only in order to prove an algebra is unital.

A **minimal** (or **primitive**) idempotent is one which is minimal among nonzero idempotents relative to the ordering; by (1.1) this is equivalent to the condition that eAe contain no idempotents $f \neq 0, e$ (since $f \in eAe$ iff $ef = fe = f$). We do not count 0 as a minimal idempotent.

From the point of view of structure theory the kind of idempotents we like best are the **division idempotents** (or **completely primitive** idempotents), those idempotents e for which eAe is a division algebra. Since division algebras contain no proper idempotents (the only invertible idempotent is 1), division idempotents are always minimal: completely primitive $\Rightarrow$ primitive. The Minimal Quadratic Ideal Theorem IV. 5.3 furnishes a host of division idempotents.

1.3 Example. In an associative matrix algebra $M_n(\Delta)$ for a division ring, the diagonal idempotents e_{ii} are division idempotents. If D is not a division algebra but still has no proper idempotents (for example, if N is a nil ideal in D with $D/N \cong \Delta$) then in $M_n(D)$ the e_{ii} are still minimal but no longer division idempotents. ∎

1.4 Example. In a Cayley algebra over a field Φ, e is a proper idempotent iff $t(e) = 1$ and $n(e) = 0$, in which case $e\mathfrak{C}e = \Phi e$, so all proper idempotents are division idempotents. (See VII. 4.21). ■

1.5 Example. Having idempotents is good, but one can have too much of a good thing. An alternative algebra is Boolean if all its elements are idempotent, $x^2 = x$ for all x. You have undoubtedly already seen examples of Boolean associative algebras. Actually, every Boolean alternative algebra is automatically associative:

1.6 (Boolean Theorem) A Boolean alternative algebra is commutative and associative.

Proof. Linearizing $x^2 = x$ gives $xy + yx = (x+y)^2 - x^2 - y^2 = (x+y) - x - y = 0$; in particular, $2x = 2x^2 = xx + xx = 0$, so A has characteristic 2 ($2A = 0$). Then $xy = - yx = + yx$ so A is commutative.

We have seen in II. 4.1 that a commutative alternative algebra is associative if $\frac{1}{3} \in \Phi$ or if there are no nilpotent elements; both of these conditions are met in our case, so A is associative. We can also argue directly: because $x^2 = x$ and $x \circ A = 0$ for all x, by Middle Bumping all associators $[x,y,z] = [z^2,y,z] = x \circ [x,y,z] = 0$ vanish. ■

VI. 1.1 Problem Set on Primitive Idempotents

We want to investigate the subalgebra eAe when e is a primitive idempotent.

1. If z is nilpotent, show no multiple az can be invertible. If z, w are nilpotent show $z+w \neq 1$ (but give an example to show $z+w$ can be invertible).
2. Prove the Proposition. If A is an alternative algebra in which each element is either invertible or nilpotent, then the set $Z = \{z \in A | z \text{ is nilpotent}\} = \{z \in A | z \text{ is not invertible}\}$ forms an ideal in A, and A/Z is a division algebra.

Recall that an algebra A in which the non-units form an ideal N is called a **local** algebra; if N is nil, A is called **strongly local**. The proposition says that an algebra with only invertible or nilpotent elements is strongly local.

3. Show that if 1 is a primitive idempotent in an algebraic alternative algebra A over a field, then each element of A is either invertible or nilpotent, so A is strongly local.
4. Prove the Proposition. If e is a primitive idempotent in an algebraic alternative algebra A over an algebraically closed field Φ, then $eAe = \Phi e + Z$ for Z a nil ideal in eAe.

VI. 1.2 Problem Set: Two Lemmas for Idempotents

Roughly, we want to show that if xy is idempotent so is yx, and if $x(yz)$ is idempotent so is $(xy)z$. This can't be quite right; even in associative algebras xy idempotent can't force yx idempotent.

1. In an associative matrix algebra find x, y with xy idempotent but not yx. (It is cheating to choose $xy = 0$!)
2. Show that if $xy = e$ is idempotent there is another idempotent f and elements x', y' with $x'y' = e$, $y'x' = f$ (even $x'y = xy' = e$, $yx' = y'x = f$).
3. (xy, yx Lemma for Idempotents) If $xy = e$ is an idempotent and either $ex = x$ or $ye = y$, then $yx = f$ is also an idempotent. (A typical example of the above would be $x = e_{12}$, $y = e_{21}$ in a matrix algebra, $xy = e_{11}$ and $yx = e_{22}$).
4. In a split Cayley algebra, find x,y,z with $x(yz)$ idempotent but not $(xy)z$.
5. Show that if $x(yz) = e$ is idempotent then there is another idempotent f and an element x' with $x'(yz) = e$, $(x'y)z = f$.
6. ($x(yz)$,$(xy)z$ Lemma for Idempotents) If $x(yz) = e$ is idempotent and $ex = x$ then $(xy)z = f$ is also an idempotent. (A typical example would be $x = e_{12}^{(1)}$, $y = e_{12}^{(2)}$, $z = e_{12}^{(3)}$ Cayley matrix units, so $x(yz) = e_{12}^{(1)}(e_{12}^{(2)}e_{12}^{(3)}) = e_{12}^{(1)}e_{21}^{(1)} = e_{11}$, $(xy)z = e_{21}^{(3)}e_{12}^{(3)} = e_{22}$).

VI. 1.3 Problem Set on Zorn Algebra

As in the associative case, an alternative algebra A is called a (left) **Zorn algebra** if for each $x \in A$ either x is nilpotent or there is an idempotent left multiple $e = yx \neq 0$. Thus except for the nil part, a Zorn algebra is full of idempotents.

1. From the associative case, or from the xy, yx Lemma for Idempotents, show that left Zorn and right Zorn are equivalent, so one speaks simply of **Zorn algebras.**
2. Show that in a Zorn algebra any element x is either properly nilpotent or else some multiple $yx = e \neq 0$ is idempotent.
3. Show that in a Zorn algebra the Jacobson and nil radicals coincide, Rad A = $\{z | z$ is properly nilpotent$\}$.
4. Show that if $xz = e$ for z nilpotent, e a central idempotent, then $e = 0$. Conclude that if all idempotents e in a Zorn algebra A are central then Rad A consists of all nilpotent elements.
5. Show that if a Zorn algebra A is **primary** (the only idempotents are 1 and 0), then A is a strongly local algebra: the non-units are the nilpotent elements, and form the unique maximal ideal.
6. Any ideal $B \triangleleft A$ in a Zorn algebra A is itself a Zorn algebra. If A is Zorn and B is a nil ideal, A/B is still

Zorn. (Even in the associative case, a general quotient A/B need not again be Zorn.)

7. If e is an idempotent in a Zorn algebra A, show eAe is Zorn.

8. Use #3 to show directly Rad(B) = B $\cap$ Rad(A) for any ideal B and Rad(eAe) = eAe $\cap$ Rad(A) for any idempotent e in a Zorn algebra A. Conclude that if A is semisimple Zorn so is any ideal B or Peirce subalgebra eAe. (See Chapter V for radicals.)

9. If e is a maximal (= principal) idempotent in a semisimple Zorn algebra, show e = 1.

VI. 1.4 Problem Set on Regular Algebras

Recall that an element x is **regular** if $x = xyx$ for some y. Algebras in which all elements are regular are rich in idempotents.

1. If A contains no nilpotent elements, show every idempotent lies in the center C(A).
2. If all idempotents of A lie in the center, show that if $x \in A$ is regular there is $y \in A$ with $xy = yx = e$ for some idempotent e with $ex = xe = x$; in this case show $Ax = Ae = eA = xA$ is a two-sided ideal. If A is regular with all idempotents central, show all one-sided ideals are two-sided.
3. If A is regular with unit 1 which is a primitive idempotent show A is a division algebra.
4. Suppose A is regular and all idempotents lie in the center. Show for each $x \neq 0$ there is a maximal (left = two-sided) ideal M_x missing x, and A/M_x is a division algebra.
5. Prove the <u>Theorem of Forsythe-McCoy</u>: A regular alternative algebra without nilpotent elements is a subdirect sum of division algebras.
6. If for each $x \in A$ there is an integer $n = n(x) > 1$ with $x^{n(x)} = x$, show A is regular without nilpotent elements.
7. Prove the <u>Jacobson Commutativity Theorem for Alternative Algebras.</u> Any alternative algebra each of whose elements

x satisfies a relation $x^{n(x)} = x$ for some $n(x) > 1$ is commutative and associative (indeed a subdirect sum of fields).

VI. 1.5 Problem Set on Kinds of Regularity

An element $x \in A$ is **regular** if $x = xyx$ for some y, π - **regular** (π for power) if $x^n = x^n y x^n$ for some n and some y (that is, if some power is regular), and **strongly (left) regular** if $x = x^2 y$ for some y. An algebra is **regular** (resp. π - **regular, strongly regular**) if all its elements are.

1. Show a (left) strongly regular element cannot be nilpotent (can it be quasi-invertible?), so a strongly regular algebra contains no nilpotents. Show that when A contains no nilpotents $x = x^2 y$ implies $x = xyx = yx^2$, so left and right strong regularity are equivalent, and both imply ordinary (weak) regularity.

2. If A has dcc on all subspaces $a^n \Phi[a] a^n$ or all $a^n A a^n$ (in particular, if A is algebraic over a field or weakly Artinian) then it is π-regular.

3. Conclude Strong Regularity $\Rightarrow$ Regularity $\Rightarrow$ π-Regularity $\Rightarrow$ Zornity.

4. Show the following are equivalent: (i) A is strongly regular, (ii) A is regular without nilpotents, (iii) A is regular with all idempotents central, (iv) A is π-regular and semisimple with all idempotents central.

5. Show that if $x \in A$ is regular (resp. π-regular, strongly regular) so is any homomorphic image of x. If x is regular (etc.) in A and lies in an ideal B or subalgebra eAe, show

x is regular (etc.) in B or eAe. Conclude that if A is regular (resp. π-regular, strongly regular), so is every homomorphic image F(A), every ideal B $\triangleleft$ A, and every Peirce subalgebra eAe.

6. Show that a central element is regular iff it is strongly regular. Show that if a central element $c \in C(A)$ is regular in A it is regular in the center C(A). Conclude that the center of a regular (resp. π-regular, strongly regular) algebra is again of the same type.

7. Prove <u>McCoy's Lemma</u> for alternative algebras: if $x - xyx$ is regular then x itself is regular, and if $x - x^2yx^2$ is strongly regular then x itself is strongly regular.

8. Show that the various types of regularity are radical properties, so every algebra contains a maximal p-regular ideal p(A), such that A/p(A) contains no p-regular ideals ($p = \phi$, π, or strongly).