Chapter V

Ideals

§1 Products of ideals

In any nonassociative algebra A the sum B + C of two ideals is again an ideal, but in general the product BC of two ideals (consisting of all finite sums $\Sigma\, b_i c_i$ of products of $b_i \in B$ with $c_i \in C$, not just of products bc) is not an ideal. The trouble is that because of nonassociativity in a product A(BC) there is no way to slip the A inside the parentheses (once inside we could use $AB \subset B$, $AC \subset C$) .

It is important that the alternative laws provide enough associativity to guarantee the product of two ideals is again an ideal.

1.1 (Product Theorem for Ideals). If B,C are ideals in an alternative algebra A then so is their product BC.

Proof. By symmetry we need only check BC is a left ideal, $A(BC) \subset BC$. But A(BC) is spanned by elements of the form a(bc) for $a \in A$, $b \in B$, $c \in C$, and $a(bc) = -b(ac) + (ab + ba)c = bc' + b'c \in BC$ for $c' = -ac \in C$, $b' = ab + ba \in B$. (Note we need B-two sided, C left in order that BC be two-sided; we will see the product of two left ideals is not always a left ideal). ◻

One simple but highly important example of a product is the square

$$D(B) = B^2 = BB \qquad (1.2)$$

of an ideal B. We can iterate this procedure to get the higher derived ideals $D^0(B) = B$, $D^{n+1}(B) = D(D^n(B))$; these are ideals in A if B is. An ideal is <u>trivial</u> if $D(B) = B^2 = 0$, i.e. all products involving two factors from B are zero. An ideal is <u>solvable</u> if one of the higher derived ideals $D^n(B) = 0$. This generalization of triviality will be important when we discuss radicals in Chapter VI.

The <u>left annihilator</u> $Ann_L(S)$ of any subset S of a linear algebra A is the set of elements in A which kill S from the left:

$$Ann_L(S) = \{a \in A \mid aS = 0\} .$$

Similarly the <u>right annihilator</u> $Ann_R(S)$ consists of the elements which kill S from the right, $Sa = 0$. (They can also be written as $S^{\perp,L}$ and $S^{\perp,R}$) . These are automatically linear subspaces, but even in the alternative case they need not be one-sided ideals. However, if the set S happens to be an ideal then so are its annihilators:

1.3 (Annihilator Lemma). If B is an ideal in A, so are its left and right annihilators $Ann_L(B)$ and $Ann_R(B)$.

Proof. We consider only the left annihilator. If $x \in Ann_L(B)$ kills B from the left, so do any ax or xa since $(ax)B = [a,x,B] + a(xB) = [x,B,a] = (xB)a - x(Ba) \subset 0 - xB = 0$ (since B is right) and $(xa)B = [x,a,B] + x(aB) \subset [x,B,a] + xB$ (since B is left) $= 0$ as above. ∎

Also useful, because the Jordan operations are so well-behaved in an alternative algebra, is the

1.4 (Jordan Product Theorem for Ideals). If B, C are ideals in an alternative algebra A so is then Jordan product $U_B C$.

Proof. It is enough if $U_B C$ is a left ideal. But $U_B C$ is spanned by elements of the form $U_b c$ for $b \in C$, $c \in C$ and $a(U_b c) = \{(ab)c\}b$ (Right Moufang) $= U_{ab,b}\, c - (bc)(ab)$ $= U_{ab,b}\, c - U_b(ca)$ (Middle Moufang) $= U_{b',b}\, c + U_b\, c' \in U_B C$ where $b' = ab \in B$, $c' = -ca \in C$. ☐

Two important and Jordan products of ideals are

$$(1.5) \qquad D^+(B) = U_B B$$
$$P^+(B) = U_B \hat{A} .$$

$D^+(B)$ is sort of the "Jordan cube" of B, while $P^+(B)$ is sort of the "Jordan square." Although $D^+(B)$ is intrinsically determined by B, $P^+(B)$ depends on the enveloping algebra A. The Jordan Product Theorem shows immediately that $D^+(B)$ is an ideal in A if B is. To see $P^+(B)$ is an ideal in A if B is, first note that an ideal $B \subset A$ remains an ideal in the unital hull $\hat{A} = \phi 1 + A$ (since B is trivially invariant under multiplication by the added part $\phi 1$), so the Jordan Product Theorem shows $U_B \hat{A}$ is an ideal in $\hat{A}$. But $U_B \hat{A}$ is contained back in A (since B is, and A is an ideal in $\hat{A}$), so $U_B \hat{A}$ is an A-ideal. We can iterate these constructions to get higher Jordan squares and cubes $P^{+n}(B)$ and $D^{+n}(B)$.

$P^+(B)$ is just the ideal generated by all squares b^2 for $b \in B$: it certainly is an ideal containing such squares, and it is generated by them because any $U_b a = bab$ can be written as $b\circ ab - ab^2 = b\circ b' - ab^2$ where $b' = ab$ also lies in B.

We thus have two squares of an ideal, the ordinary square $D(B) = B^2$ and the Jordan square $P^+(B) = U_B\hat{A}$, as well as a Jordan cube $D^+(B) = D_B B$. These are related as follows.

1.6 (Two-squares-and-a-cube Lemma.) For any ideal B in an alternative algebra A we have natural inclusions

$$D^+(B) \subset P^+(B) \subset D(B)$$

as well as relations

$$P^{+2}(B) \subset D(P^+(B)) \subset D^+(B) \qquad 2D^2(B) \subset P^+(B)$$

Further, $D(B) = B^2 = \Sigma\, Bb$ where each $Bb + P^+(B)$ is an ideal trivial modulo $P^+(B)$.

Proof. The natural inclusions are $U_B B \subset U_B\hat{A} \subset B(\hat{A}B) = B^2$. For the unnatural inclusions, $P^+(P^+(B)) \subset D(P^+(B))$ since always $P^+ \subset D$. For $D(P^+(B)) \subset D^+(B)$ observe that $P^+(B)$ is spanned by elements bxb $(b \in B,\ x \in \hat{A})$, so $D(P^+(B)) = P^+(B)^2$ is spanned by $(bxb)(cyc)\ (b,c \in B, x,y \in \hat{A})$, where mod $D^+(B) = U_B B$

$$\begin{aligned}
([bx]b)(c[yc]) &= U_{bx,yc}(bc) - ([yc]b)(c[bx]) && \text{(Middle Moufang)}\\
&\equiv -([yc]b(c[bx])) && (bx, yc, bc \in B)\\
&= -y\cdot U_{c,c[bx]}b + \{(y\cdot c[bx])b\}c && \text{(Right Moufang)}\\
&\equiv \{(y\cdot c[bx])b\}c && (c, b, c[bx] \in B)\\
&= \{U_{y,b}\, c[bx]\}c - \{(b\cdot c[bx])\, y\}c &&\\
&= y\{c[bx](bc)\} + b\{c[bx](yc)\} - \{([bcb]x)\, y\}c && \text{(Left Moufang)}\\
&\equiv y\cdot U_c[bx]b + b\cdot U_c[bx]y && (\text{Middle Moufang}; b,c \in B)\\
&\equiv 0 && ((bx)b, (bx)y,\ c \in B).
\end{aligned}$$

To see $2D^2(B) \subset P^+(B)$ observe that by Middle Moufang $U_b(xy) \equiv 0 \mod P^+(B)$, so

(1.7) $$(bx)(yb) \equiv 0 \quad (bx)(yc) \equiv -(cx)(yb) \quad (b,c \in B).$$

Since also $b \circ c = U_{b,c}1 \equiv 0$ we have

(1.8) $$bc \equiv -cb.$$

Thus $(bb')(c'c) \overset{(1.7)}{\equiv} -(cb')(c'b) \overset{(1.8)}{\equiv} -(b'c)(bc') \overset{(1.7)}{\equiv} +(c'c)(bb')$ $\overset{(1.8)}{\equiv} -(bb')(c'c)$ (since $bb', c'c \in B$ have $bb' \circ c'c \equiv 0$), or $2(bb')(c'c) \equiv 0$. Thus $2D^2(B) = 2\{BB\}\{BB\} \subset P^+(B)$.

To prove $Bb + P^+(B)$ is an ideal trivial mod $P^+(B)$, it suffices to divide out by $P^+(B)$ and prove $\bar{B}\bar{b}$ is a trivial ideal in $\bar{A} = A/P^+(B)$. Since b^2 and $\{b\,\hat{A}\,B\}$ lie in $P^+(B)$ this follows from

1.9 Lemma. If $z^2 = U_{z,B}\hat{A} = 0$ for $B \triangleleft A$ then $zB = Bz$ is a trivial ideal in A.

Proof. $zB = -Bz$ since $z \circ B = U_{z,B}1 = 0$ by hypothesis. The subspace is trivial since $(zB)^2 = (zB)(Bz) = U_z B^2$ (Middle Moufang) and $U_z b = z(b \circ z) - z^2 b = 0$ if $B \circ z = z^2 = 0$. It is a left ideal since $A(Bz) = U_{A,z}B - z(BA) = (A \circ B)\circ z - U_{z,B}A - z(BA) \subset B \circ z - 0 + zB = zB$ (using $U_{z,B}\hat{A} = 0$). Similarly it is a right ideal. ∎∎

1.10 Corollary. If $P^+(B) = U_B\hat{A} = 0$ then all Bb are trivial ideals in A. ∎

These results will help us compare the various notions of solvability that go with the various notions of higher powers.

We have isolated Lemma 1.9 from the proof for future reference (see Problem Set #00).

1.11 (Minimal Ideal Theorem). A minimal ideal B in an alternative algebra is either trivial or simple as an algebra.

Proof. We consider two cases, according as $D^+(B) = U_B B$ is B or 0 (these are the only two possibilities, since $D^+(B)$ is an ideal of A contained in B and B is minimal).

If $D^+(B) = 0$ we will show B is trivial, $B^2 = 0$. By the Two-squares-and-a-cube Lemma, $P^{+2}(B) \subset D^+(B) = 0$; if $P^+(B) \neq 0$ then $P^+(B) = B$ (as an ideal contained in B), which would imply $P^{+2}(B) = P^+(P^+(B)) = P^+(B) = B$, contrary to the above. Therefore $P^+(B) = 0$. But then by Corollary 1.10 all Bb are trivial ideals in A. As they are contained in B, either Bb = B for some b (in which case B is trivial; since Bb is; better yet, Bb = B implies $B = Bb = (Bb)b = Bb^2 = 0$ since $b^2 \in P^+(B) = 0$), or else Bb = 0 for all b (in which case again BB = 0). Thus $B^2 = 0$ when $D^+(B) = 0$.

Now assume $D^+(B) = B$. (In particular, $B^2 = B$ since $D^+(B) \subset B^2 \subset B$). In this case B will turn out to be simple. We begin by showing that if $C \lhd B$ then $B(CB) \lhd A$. Since $B = D^+(B) = U_B B$, $L_A L_B$ is spanned by operators $L_a L_{bdb} = L_a L_b L_d L_b$ (left Moufang, for $b, d \in B$) $= \{L_a L_b L_d + L_d L_b L_a\} L_b - L_d L_b L_a L_b$ $= L_{U(a,d)b} L_b - L_d L_{bab} \in L_B L_B$. Thus $A\{B(CB)\} = L_A L_B(CB) \subset L_B L_B(CB) = L_B\{B(CB)\} \subset B(CB)$ (using the Product Theorem to see

B(CB) is an ideal in the algebra B). This makes B(CB) a left A-ideal. It is a right A-ideal since $\{B(CB)\}A = [B,CB,A] + B\{(CB)\,A\}$ where $[A,B,CB] = (AB)(CB) - A\{B(CB)\} \subset B(CB)$ by the previous result, and where $(CB)A \subset CB$ since again $R_a\,R_{bdb} = R_aR_bR_dR_b = R_{U(a,d)b}\,R_b - R_d\,R_{bab}$ shows $R_AR_B \subset R_BR_B$ with CB a B-ideal. Once B(CB) is a (two-sided) A-ideal we have by minimality of B either $B(CB) = B$ (whence $B = B(CB) \subset C$ if $C \triangleleft B$, so $B = C$) or else $B(CB) = 0$ (whence $CB = 0$ since $B^2 \neq 0 \Rightarrow B \cap \mathrm{Ann}_R\, B \neq B \Rightarrow B \cap \mathrm{Ann}_R\, B = 0$, then similarly $C = 0$ since $C \subset B \cap \mathrm{Ann}_L\, B = 0$). We have shown that the only ideals C in B are $C = B$ or $C = 0$, and $B^2 \neq 0$, so B is simple as an algebra. ∎

Exercises

1.1 If B,C,D are ideals in an alternative algebras A, show the Jordan product $U_{B,C}\,D$ (spanned by all elements $U_{b,c}\,d$) is again an ideal.

1.2 Verify directly that if B,C are ideals so are the left and right transporters $L(B,C) = \{x \in A \mid xB \subset C\}$ and $R(B,C) = \{x \in A \mid Bx \subset C\}$. Conclude $\mathrm{Ann}_L(B)$ and $\mathrm{Ann}_R(B)$ are ideals.

1.3 Verify that $L(B,C) = r^{-1}(\mathrm{Ann}_L\, r(B))$ (for $r: A \to A/C$ the canonical projection), dually $R(B,C) = r^{-1}(\mathrm{Ann}_R\, r(B))$, so that the fact that annihilators are ideals implies the more general result that transporters are ideals.

← Another exercise.

1.4 Let B be an ideal in A, C an ideal of B which is <u>idempotent</u> in the sense that $C^2 = C$. Show that C is actually an ideal of A. (Usually $C \lhd B \lhd A$ does not imply $C \lhd A$).

1.5 Prove $4D^3(A) \subset D^+(A)$ by showing that modulo $D^+(A)$ the product $(x_1x_2)(x_3x_4)$ is alternating in x_1, x_4 and in x_2, x_3, then $(x^2y)z^2 \equiv 0$, then $x\{y(zw)] + (xy)(wz) \equiv 0$, then $2x^2\{y(zw)\} \equiv 0 \equiv 2\{x^2y\}(zw)$.

1.6 Show B is trivial iff $B \subset \mathrm{Ann}_L(B)$. If some $D^n(B) \subset \mathrm{Ann}_L(B)$ show B is solvable; what about the converse?

1.7 If $1/2 \in \Phi$ show B is solvable (some $D^n(B) = 0$) iff some $P^{+m}(B) = 0$.

1.8 What can you say about a minimal ideal B (in A) which is trivial as an algebra?