§7. Quasi-invertibility

The radical we will be most interested in is the Jacobson-Smiley radical, the largest quasi-invertible ideal. It can be characterized element-wise as the set of all properly quasi-invertible elements. This radical is the largest or most general we shall consider; semisimplicity of an algebra (absence of quasi-invertible ideals) implies nil-freeness, local-nilpotence-freeness, strong semiprimeness, and semiprimeness.

Once we have a notion of inverse we can introduce a notion of quasi-inverse. Recall that x is **quasi-invertible** with **quasi-inverse** y if 1-x is invertible with inverse 1-y (all this taking place in $\hat{A}$ if A is not unital). The invertibility conditions $(1-x)(1-y) = (1-y)(1-x) = 1$ are equivalent to the intrinsic conditions

$$x+y = xy = yx. \tag{7.1}$$

However, just as for associative algebras it will be conceptually and computationally simpler to think in terms of ordinary invertibility of 1-x and 1-y. For example, from the uniqueness and symmetry of ordinary inverses we deduce that the quasi-inverse y of x is unique and that symmetrically x is the quasi-inverse of y.

It is also important to note that if 1-x is invertible at all in $\hat{A}$, with inverse u, then automatically $u \in \hat{A}$ has the form $u = 1-y$ for some $y \in A$: if we define $y = 1-u$ then $1 = u(1-x) = u-ux$ implies $y = 1-u = -ux$ lies in $\hat{A}A \subset A$. Thus the invertibility of 1-x is all that matters, not the particular form of its inverse.

Exercises IV.6

6.1 If B is an ideal of A which is nil of bounded index 2, $b^2 = 0$ for all $b \in B$, show inside B there is a trivial ideal of A. Show B itself need not be nilpotent.

6.2 (Open) Can you generalize the above to the case when B is nil of bounded index n? What can you say if B is only a one-sided ideal of bounded index? [Using structure theory one can show either B contains a trivial ideal (resp. left ideal) of A, or else $3B = 0$. It would be desirable to have a direct proof which also works in characteristic 3].

6.3 If $C \lhd B \lhd A$ show the ideal I(C) generated by C in A is nil modulo C: for each $z \in I(C)$ there is an integer n such that $z^n \in C$.

6.4 If C is a left ideal in A, show the ideal I(C) (namely its hull H(C)) it generates is nil modulo C.

6.5 Use Anderson-Divinsky-Sulinsky to show that when $C \lhd B \lhd A$ then $I(C) = \Sigma C_\alpha$ is a sum of B-ideals $C_\alpha \lhd B$ which are solvable mod C. Conclude I(C) is locally nilpotent mod C.

6.6 If z is trivial modulo the nil radical, $U_z A \subset \mathrm{Nil}(A)$, show z belongs to Nil(A).

6.7 Prove the <u>Nil Radical Inheritance Theorem</u>: The nil radical of an ideal B in an alternative algebra A is

$$\mathrm{Nil}(B) = B \cap \mathrm{Nil}(A).$$

6.8 Prove the <u>Nil-free Inheritance Theorem</u>: If A is a nil-free alternative algebra so is any ideal $B \lhd A$.

6.9 If Φ contains no nilpotent elements show $\mathrm{Nil}(\hat{A}) = \mathrm{Nil}(A)$; in general show $\mathrm{Nil}(\hat{A}) = \{\mathrm{Nil}(\Phi)\}1 + \mathrm{Nil}(A)$.

6.10 If A is nil free alternative, is every Peirce subalgebra eAe ($e \in \hat{A}$ idempotent) also nil free?

IV.6.1 Problem Set on Proper Nilpotence

1. Prove the <u>xy,yx-Lemma for Nilpotence</u>: An element xy is nilpotent iff yx is nilpotent.

2. Prove the <u>(xy)z,x(yz)-Lemma for Nilpotence</u>: An element x(yz) is nilpotent iff (xy)z is nilpotent.

3. We say z is properly nilpotent (p.n.) if all multiples az are nilpotent. Show this is equivalent to all za being nilpotent. Show that if z is p.n. so are all az and za. Show all trivial elements are p.n.

4. It is not clear even in the associative case that the sum of two p.n. elements is again p.n. (this is one version of the Koethe conjecture). Show however that if z is p.n. and w is trivial then z+w is p.n.

5. If PN(A) denotes the set of properly nilpotent elements of A, show Nil(A) $\subset$ PN(A) and that they coincide if PN(A) is closed under sums. Show any nil one-sided ideal is contained in PN(A); conclude that if Nil(A) = PN(A) then Koethe's conjecture holds for A.

IV.6.2 Problem Set on Nil-implies-nilpotence

We wish to show that if A has a.c.c. on subalgebras (e.g., if it is finite-dimensional over a field) then the nil radical is nilpotent.

1. If B is a proper subalgebra of A with $M_A(B)$ nilpotent, show there is an element $c \in A$, $c \notin B$ with $M(B)c \subset B$. Conclude $C = B + \Phi[c]$ is a subalgebra properly containing B as an ideal.
2. If the element c of Exercise 1 is nilpotent, show $M_A(C)$ is nilpotent (show $\hat{M}_E(C)M_E(B) \subset M_E(B)\hat{M}_E(C)$ whenever $B \triangleleft C$ and E is an enveloping algebra $E \supset C$, and deduce $\{\hat{M}_E(C)M_E(B)\hat{M}_E(C)\}^n \subset M_E(B)^n\hat{M}_E(C))$.
3. If B is a proper subalgebra which acts nilpotently on A (i.e., $M_A(B)$ acts nilpotently) where A is a nil alternative algebra, show there exists a larger proper algebra $B < C \subseteq A$ still having $M_A(C)$ nilpotent.
4. If A is a nil alternative algebra with a.c.c. on subalgebras, deduce that A is nilpotent.
5. Deduce the <u>Theorem</u> If A is a nil alternative algebra with a.c.c. on subalgebras, then any nil subalgebra is nilpotent. In particular, Nil(A) is nilpotent.
6. Conclude that if A is finite-dimensional over a field, or finitely spanned over a field Φ, then every nil subalgebra is nilpotent.

Henceforth we will abbreviate "quasi-invertible" by "q.i.". A subalgebra B will be called q.i. if all its elements are q.i. in B; in general this is not the same as being q.i. in A, for an element b may well have a quasi-inverse c in A which happens not to lie in B. This can't happen if B is an ideal, or more generally

7.2 (Q.I. Closure Proposition) A strict quadratic ideal B in an alternative algebra A is quasi-inverse closed: if $b \in B$ has a quasi-inverse $c \in A$ then necessarily $c \in B$.

Proof. From (7.1) $b+c = bc = cb$ we see $c = bc-b = b(cb-b)-b = bcb-b^2-b$ lies in B since $U_b A \subset B$ if B is a quadratic ideal, and $b^2 \in B$ if B is strict. ∎

7.3 Corollary. A strict quadratic ideal is q.i. iff it is q.i. in A. ∎

Since one-sided ideals are strict quadratic ideals,

7.4 Corollary. If a (left, right, or two-sided) ideal is q.i. in A it is q.i. in itself. ∎

Thus the condition that the elements of an ideal B have quasi-inverses in A is equivalent to the intrinsic condition that B, as an algebra in its own right, is q.i. Once this observation is made we can establish

7.5 (Radical Theorem for Q.I.) Quasi-invertibility is a strongly hereditary radical property: (i) if an alternative algebra A is q.i. so is any homomorphic image and any one-sided ideal in A, (ii) if A/B and B are q.i. so is A, (iii) the union of a chain of q.i. ideals is again q.i. Therefore each alternative algebra A contains a largest q.i. ideal Rad(A), which is also the smallest ideal whose quotient is free of q.i. ideals.

Proof. (i): If x and y are quasi-inverses in A, $x+y = xy = yx$, then $F(x)$ and $F(y)$ are quasi-inverses in a homorphic image $F(A)$, $F(x)+F(y) = F(y)F(x)$. We just observed that any one-sided ideal in a q.i. algebra A is itself q.i.

(ii): If $\bar{A} = A/B$ and B are q.i. then for any $x \in A$ the image $\bar{x}$ in $\bar{A}$ is q.i., so $\bar{x}+\bar{z} = \overline{xz} = \overline{zx}$ or $(\bar{1}-\bar{x})(\bar{1}-\bar{z}) = (\bar{1}-\bar{z})(\bar{1}-\bar{x}) = \bar{1}$ for some $z \in A$. Then $(\bar{1}-\bar{x})(\bar{1}-\bar{z})^2(\bar{1}-\bar{x}) = \bar{1}$ or $(1-x)(1-z)^2(1-x) = 1-b$ for some $b \in B$; since B is q.i. the element $1-b = U_{1-x}(1-z)^2$ is invertible in A, which forces $1-x$ to be invertible (by I.4.5 $U_a c$ invertible implies a and c are invertible). Consequently x is q.i.

(iii): Since q.i. is an element-condition, any union of q.i. ideals forms a set of q.i. elements in A, so $B = \cup B_\alpha$ is an ideal and is q.i. in A, therefore is q.i. in itself.

Thus A always contains a maximal q.i. ideal containing all other q.i. ideals, namely the sum or union of all possible q.i. ideals. The recoverability (ii) guarantees as usual that Rad(A) is the smallest ideal R such that A/R is free of q.i. ideals. ■

This maximal q.i. ideal is called the *Jacobson-Smiley radical* Rad(A). The condition that its quotient be free of q.i. ideals is

$$\mathrm{Rad}(A/\mathrm{Rad}\ A) = 0.$$

An algebra is *semisimple* if it has no radical, Rad A = 0. Thus A/Rad A is always semisimple.

We have observed several times that a nilpotent element is q.i., $(1-z)^{-1} = 1+z+\cdots+z^{n-1}$ if $z^n = 0$. Therefore any nil ideal is q.i.; in particular, the nil radical is contained in the Jacobson-Smiley-radical, giving us by (6.2) the latest chain of inclusions

(7.6) $$S(A) \subset T(A) \subset L(A) \subset Nil(A) \subset Rad(A).$$

Therefore the absence of quasi-invertibility implies the absence of nilness, triviality, and nilpotence:

(7.7) semisimple $\Rightarrow$ nil-free $\Rightarrow$ local-nilpotence free $\Rightarrow$ strongly semiprime $\Rightarrow$ semiprime.

For our purposes, the Jacobson-Smiley radical is <u>the</u> radical. We denote it simply by Rad(A), rather than JS(A), and whenever we speak of "the radical" (without any adjectives) it is always the Jacobson-Smiley radical we have in mind.

Since ideals are so troublesome to construct, it will be much easier if we can find a way to describe the radical directly in terms of its elements. Judging by the associative case, we expect the radical to consist precisely of the properly quasi-invertible elements.

Before we start we need to collect some operator identities which will help us over the hurdle of nonassociativity. Since Jordan products are so well-behaved we deal with Jordan operators. We introduce the so-called transvections

(7.8) $$T_{x,y} = I-V_{x,y} + U_xU_y \quad (V_{x,y}z = U_{x,z}y).$$

The reason transvections have something to do with quasi-invertibility is because invertibility of 1-x amounts to invertibility of U_{1-x}, and

(7.9) $$T_{1,x} = T_{x,1} = I-V_x + U_x = U_{1-x} .$$

(In Section 8 we will see that invertibility of $T_{x,y}$ amounts to quasi-invertibility of x in the y-homotope $A^{(y)}$). They satisfy the transvection associativity formula

(7.10) $$R_yT_{x,yz} = T_{xy,z}R_y,$$

as is verified by computing $R_yT_{x,yz}a = R_y\{a-U_{x,a}yz+U_xU_{yz}a\} = R_ya-R_yU_{x,a}L_yz + R_yU_xL_yU_zR_y = ay-U_{xy,ay}z+U_{xy}U_z(ay)$ (Left and Right Fundamental Formula and its linearization) $= \{I-V_{xy,z}+U_{xy}U_z\}(ay) = T_{xy,z}R_ya$.

Transvections provide a technical criterion for quasi-invertibility of a product.

7.11 (Lemma) xy is q.i. iff U_xy-2x lies in the image $T_{x,y}(\hat{A})$.

Proof. If xy is q.i. then by (7.9) $T_{1,xy}$ is invertible on $\hat{A}$, hence $R_x\{xy-2\} = U_xy-2x$ is in the image of $R_xT_{1,xy} = T_{x,y}R_x$ (substitute $x \to 1$, $y \to x$, $z \to y$ in (7.10)). Conversely, if U_xy-2x is in the image $T_{x,y}(\hat{A})$ then $R_y\{U_xy-2x\} = (xy)^2-2xy$ is in the image of $R_yT_{x,y} = T_{xy,1}R_y$ (set $z = 1$ in (7.10)); but $T_{xy,1}1 = U_{1-xy}1 = 1-2xy+(xy)^2$ is already in the image of $T_{xy,1}$, so subtracting puts 1 in the image of $T_{xy,1} = U_{1-xy}$ and hence xy is q.i. by the Inverse Theorem I.4.2vi. ■

The next two lemmas allow us to act as though we were working in a commutative and associative situation.

7.12 (xy,yx-Lemma for Quasi-invertibility) An element xy is q.i. iff yx is q.i.

Proof. By symmetry it suffices to show yx q.i. $\Rightarrow$ xy q.i. If $1-yx$ is invertible we claim $1-xy$ has inverse

$$(1-xy)^{-1} = 1+x(1-yx)^{-1}y.$$

If B is a maximal associative subalgebra containing x and y we know B is quasi-inverse closed (by the Quasi-Inverse Closure Theorem II.3.16), so x,y, and

$(1-yx)^{-1}$ belong to B. Since everything takes place in the associative algebra B, we can dispense with parentheses. We show the right side acts as a right inverse for 1-xy:

$$(1-xy)\{1+x(1-yx)^{-1}y\} = 1+x(1-yx)^{-1}y-xy-xyx(1-yx)^{-1}y$$

$$= 1-xy+x(1-yx)(1-yx)^{-1}y$$

$$= 1-xy+xy = 1.$$

Similarly it acts as a left inverse, so we have exhibited an inverse for 1-xy and xy is q.i. ∎

7.13 Remark. There is a very useful heuristic device for arriving at such strange expressions for quasi-inverses, which consists simply of writing

$$(1-a)^{-1} = 1+a+a^2 +\cdots = \sum_{k=0}^{\infty} a^k.$$

Of course this is seldom literally true (it is if a is nilpotent or there is a suitable topology around), but it is always suggestive. For example, to find $(1-xy)^{-1}$ we write

$$(1-xy)^{-1} = 1+xy+(xy)^2 +\cdots+ (xy)^n +\cdots$$

$$= 1+x\{1+yx +\cdots+ (yx)^{n-1} +\cdots \}y$$

$$= 1+x\{1-yx\}^{-1}y.$$

and discover the correct answer. ∎

7.14 (x(yz),(xy)z-Lemma for Quasi-invertibility). An element x(yz) is q.i. iff (xy)z is q.i.

Proof. It suffices to prove one direction, [redacted] x(yz) q.i. $\Rightarrow$ (xy)z q.i. (Then in the opposite algebra $(xy)z = z\cdot(y\cdot x)$ q.i. $\Rightarrow (z\cdot y)\cdot x = x(yz)$ is q.i., noting a is q.i. in A iff it is q.i. in A^{op}).

But $x(yz)$ q.i. $\Rightarrow U_x(yz)-2x \in \text{Im } T_{x,yz}$ (by 7.11) $\Rightarrow R_y\{U_xL_yz-2x\} = U_{xy}z-2xy \in \text{Im } R_yT_{x,yz} = \text{Im } T_{xy,z}R_y$ (by Right Fundamental and (7.10)) $\Rightarrow (xy)z$ q.i. (by (7.11)). ∎

Now we can establish our main result. As in the associative case we call an element z **properly quasi-invertible** (abbreviated p.q.i.) if all multiples az for $a \in A$ are q.i. By the xy,yx-Lemma this is equivalent to all za being q.i.

As usual, our terminology is justified by the fact that a p.q.i. element z is in particular q.i.: this is trivial if $1 \in A$, but anyway $z^2 = zz$ is q.i. if z is p.q.i., and in general z^n q.i. implies z q.i. (If $1-z^n = (1-z)(1+z+\cdots+z^{n-1}) = (1+z+\cdots+z^{n-1})(1-z)$ is invertible so is $1-z$. Observe that the converse is false! If $\frac{1}{2} \in \Phi$ then $z = -1$ is q.i. since $1-z = 2$, but $z^2 = 1$ is never q.i.). Thus a p.q.i. element is one which is q.i. and stays q.i. when you multiply it by anything.

7.15 Example. Any element of a q.i. one-sided ideal is automatically p.q.i.: if B is a q.i. left ideal and $z \in B$ then all az are still in B and hence still q.i. (For a right ideal all za are q.i.). ∎

The multiples $\hat{a}z$ of a p.q.i. element z not only remain q.i., they remain p.q.i.: for any $\hat{a} \in \hat{A}$ and $b \in A$ we know $(b\hat{a})z$ is q.i. when z is p.q.i. since $b\hat{a} \in A$, therefore by the (xy)z, x(yz)-Lemma $b(\hat{a}z)$ is q.i. for all b, so $\hat{a}z$ is p.q.i. Similarly $z\hat{a}$ is p.q.i. since all $(z\hat{a})b$ are q.i.

The fact that the x(yz), (xy)z-Lemma allows us to disregard nonassociativity is the key to the elementwise characterization of the radical.

7.16 (Characterization Theorem) The Jacobson-Smiley radical of an alternative algebra consists precisely of the properly quasi-invertible elements:

$$Rad(A) = PQI(A).$$

Proof. Example 7.15 shows that the q.i. ideal Rad(A) is contained in PQI(A). The reverse inclusion will follow by showing PQI(A) is an ideal, since it is automatically q.i. (p.q.i. $\Rightarrow$ q.i.) and therefore contained in the maximal q.i. ideal Rad(A).

PQI(A) is closed under scalars since z p.q.i. $\Rightarrow$ all $(\alpha a)z$ are q.i. $\Rightarrow$ all $a(\alpha z)$ are q.i. $\Rightarrow$ αz is p.q.i. We have already noted that it is closed under left and right multiplications: az and za are p.q.i. if z is. Finally, it is closed under addition: if z,w are p.q.i. their sum z+w is q.i. since $1-(z+w) = (1-z)-w = (1-z)\{1-(1-z)^{-1}w\}$ is invertible because both factors are, 1-z being invertible if z is q.i. and $1-\hat{a}w$ being invertible if w is p.q.i. The sum z+w is actually p.q.i. since any $a(z+w) = az+aw = z'+w'$ is q.i. as the sum of two p.q.i. elements $z' = az$, $w' = aw$.

Thus PQI(A) is an ideal, coinciding with Rad(A). ∎

From this it immediately follows by 7.15 that Rad(A) contains all q.i. one-sided ideals (see Ex. 7.10). Because one-sided ideals are so difficult to construct and do not arise naturally in alternative algebras, we have not been paying much attention to the radicality of one-sided ideals.

In connection with proper quasi-invertibility, note that no nonzero element can be both regular and p.q.i.: if z is regular then z = zaz for some a, and if az is q.i. then 1-az is invertible so z(1-az) = 0 implies z = 0. By customary abuse of language we can disregard zero and rephrase this as

7.17 (Radical Regularity Proposition) The radical contains no regular elements. ∎

IV.7 Exercises

7.1 Show z is q.i. in A iff $I-L_z$ is invertible on A.

7.2 Conclude x is q.i. in $A^{(y,L)}$ iff xy is q.i. in A. Dually show x is q.i. in $A^{(R,y)}$ iff yx is q.i. in A.

7.3 Show x is q.i. iff there is an element z with $U_x z = x^2+x\circ z-z-2x$.

7.4 Show a product xy is q.i. iff there is an element z with $U_xU_y z = U_x y + U_{x,z}y-z-2x$.

7.5 Use the Jordan condition of Ex. 7.3 to show x is q.i. in $A^{(y,L)}$ iff x is q.i. in $A^{(R,y)}$; deduce the xy,yx-Lemma.

7.6 Show that u,v are inverses iff

(i) $u\circ v = 2$

(ii) $U_u v = u$ (Jordan conditions for invertibility)

(iii) $U_u v^2 = 1$

Show x,y are quasi-inverses iff

(i') $x\circ y = 2x+2y$

(ii') $U_x y = x+y+x^2$ (Jordan conditions for quasi-invertibility)

(iii') $U_x y^2 = x^2-y^2+U_{x,y}y$.

Since both homotopes $A^{(u,L)}$, $A^{(R,u)}$ have the same Jordan structure, show x,y are quasi-inverses in $A^{(u)}$ iff

(i") $U_{x,y}u = 2(x+y)$

(ii") $U_xU_u y = x+y+U_x u$ (Jordan conditions for quasi-invertibility in a homotope)

(iii") $U_xU_uU_y u = U_x u-U_y u+U_{x,y}U_u y$.

Use this to show that if $x(yz)$ has quasi-inverse (necessarily of the form $w(yz)$) then $(xy)z$ has quasi-inverse $(wy)z$.

7.7 Prove directly L_{1-x} is bijective on $\hat{A}$ if it is on A.

7.8 Show L_u is invertible on A iff it is invertible on both B and A/B ($B \triangleleft A$).

7.9 Quasi-invertibility of z means 1-z is invertible. Show that if z is p.q.i. then u-z is invertible for all invertible $u \in \hat{A}$. Deduce the Sum Corollary: if z is q.i. and w is p.q.i. then z+w is q.i. Conclude that any finite sum of p.q.i. elements is p.q.i.

7.10 Prove the One-sidedness Theorem for Quasi-invertibility. Any q.i. one-sided ideal generates a q.i. two-sided ideal. Thus Rad(A) contains all q.i. one-sided ideals, and a semisimple algebra contains no q.i. one-sided ideals.

7.11 Prove the xy,yx-Lemma directly by showing that if xy is q.i. then $z+yx = z(yx) = (yx)z$ for $z = -y(1-xy)^{-1}x$.

7.12 Show z is p.q.i. in A iff all aza and zaz for $a \in A$ are q.i. (To show aza, zaz q.i. imply az q.i., reduce to the associative case and apply the Density Theorem).

IV.7.1 Problem Set on One-sided Quasi-inverses

We define one-sided quasi-inverses in the natural way: x is *left quasi-invertible* (abbreviated l.q.i.) with *left quasi-inverse* y if $1-x$ is left invertible with left inverse $1-y$ in $\hat{A}$, $(1-y)(1-x) = 1$. Right quasi-invertibility (r.q.i.) is defined similarly.

1. Find the intrinsic definition of left quasi-invertibility in A. Show that if $1-x$ has any left inverse $u \in \hat{A}$ then necessarily $u = 1-y$ for some $y \in A$.
2. Show x is q.i. iff it is both l.q.i. and r.q.i.
3. If B is a subspace all of whose elements are l.q.i. in B, show B is q.i.
4. If B is a left ideal whose elements are l.q.i. in A, show B is q.i. Conclude Rad A contains all l.q.i. left ideals and all r.q.i. right ideals.
5. Show that l.q.i. is a hereditary radical property such that the l.q.i. radical coincides with the q.i. radical Rad A.