§6. Nilness

Naturally, the nil radical is going to be the maximal nil ideal. In any power associative algebra there exists a maximal nil ideal, whose quotient is free of nil ideals; for this reason the nil radical is one of the most convenient radicals for studying power associative algebras. In the alternative case this nil radical is more general than the Levitzki radical.

An algebra or ideal is **nil** if it consists entirely of nilpotent elements, elements z with $z^n = 0$ for some n. (This makes sense in any power associative algebra). Be careful not to confuse nilness (nilpotence of the elements) with nilpotence (nilpotence of the algebra as a whole).

6.1 (Nil Radical Theorem) Nilness is a very strongly hereditary radical property in power associative algebras: (i) if A is nil so is any subalgebra or homomorphic image, (ii) if A/B and B are nil so is A, (iii) the union of a chain of nil ideals is nil. Therefore each power associative algebra A contains a largest nil ideal Nil(A), which is also the smallest ideal whose quotient has no nil ideals.

Proof. Because nilness is defined elementwise, (i) is clear. To check recoverability (ii): if $\bar{A} = A/B$ and B are nil then for any $x \in A$ the image $\bar{x} \in \bar{A}$ is nilpotent, $\bar{x}^n = 0$ for some n, so $x^n \in B$; but since B is nil $(x^n)^m = 0$ for some m, so BY POWER ASSOCIATIVITY $x^{nm} = 0$.

Since nilness is defined in terms of elements, the union of a directed set of nil ideals is again a nil ideal, so (iii) holds. Since the same argument from (i) and (ii) as in 2.6 guarantees that any finite sum of nil ideals is nil (if B is nil so is its image $B/B \cap C \cong B+C/C$ by (i), and if C is also nil so is B+C by recoverability (ii)), and the sum of all nil ideals is the union of all finite sums of nil ideals, we see that the sum or union of all nil ideals forms the unique maximal nil ideal Nil(A). Again (i) and (ii) imply Nil(A) is the smallest ideal R such that A/R is free of nil ideals: A/Nil(A) is free of nil ideals since if B/Nil(A) is nil so is B by recoverability (ii), so $B \subset$ Nil(A) by maximality and B/Nil(A) is the zero ideal, and nothing smaller will do since if $\bar{A} = A/R$ is nil-free then the nil image $\overline{Nil(A)} = Nil(A)/R$ by (i) must be zero, and $Nil(A) \subset R$. ∎

We call Nil(A) the **nil radical**; it is a nil ideal with

$$Nil(A/Nil(A)) = 0.$$

In the rest of this section we return to the setting of altennative algebras. Clearly a locally nilpotent algebra is locally nil, so globally nil. In particular, nil-free algebras are free of local nilpotence, which leads to the chain of inclusions which follows

(6.2) Nilfree ⇒ local-nilpotence free ⇒ strongly semiprime ⇒ semiprime

$$S(A) \subset T(A) \subset L(A) \subset Nil(A).$$

As in the associative theory it is still an open question (the Koethe Conjecture) whether nilness is a one-sided radical property: whether Nil(A) contains all one-sided nil ideals, i.e., whether nil one-sided ideals generate nil two-sided ideals. As in the associative case, an elementwise characterization of the nil radical is lacking (see the Problem Set).