5. Strong semiprimeness

Because ideals are so messy to construct in an alternative algebra (the ideal generated by z does not have the simple form $\hat{I}(z) = \hat{A}z\hat{A}$), it is desirable to have an element-condition for semiprimeness rather than an ideal condition. Strong semiprimeness (the absence of trivial elements) is a convenient and useful concept, which always implies semiprimeness and is equivalent to it in characteristic $\neq 3$ situations. Slinko's Theorem asserts that an algebra generated by trivial elements is locally nilpotent, so an algebra with no locally nilpotent ideals is necessarily strongly semiprime.

We say A is **strongly semiprime** if it contains no trivial elements, an element z being **trivial** if $U_zA = 0$. Before III.1.11 we noted that the existence of trivial elements was equivalent to the existence of **strictly trivial** elements $U_z\hat{A} = 0$.

A strongly semiprime algebra is certainly semiprime, for if it were to contain a trivial ideal B any element $b \in B$ would be trivial: $U_bA = b(Ab) \subset BB = 0$. In the associative case it is easy to show that a trivial element generates a trivial ideal, so a semiprime algebra is also strongly semiprime. This does not quite work in the alternative case, although trouble only occurs in characteristic 3 (which is <u>the</u> troublesome characteristic for alternative algebras).

Since we do not want to restrict ourselves to algebras over fields, by "characteristic $\neq 3$" we could mean "$\frac{1}{3} \in \Phi$", i.e., multiplication by 3 is a bijective linear transformation. However, it is actually enough if 3 is surjective <u>or</u> injective.

5.1 (Kleinfeld's Strong Semiprimeness Theorem) If A is a semiprime alternative algebra on which 3 is injective or surjective then A is strongly semiprime: if A contains no trivial ideals $B^2 = 0$ it contains no trivial elements $zAz = 0$.

Proof. Suppose A is semiprime but does contain trivial elements, hence a strictly trivial $z \neq 0$: $z\hat{A}z = 0$. In the associative case a trivial z would belong to $I(z)^{\perp} = \{w | w\hat{A}z = z\hat{A}w = 0\}$, hence $z \in I(z) \cap I(z)^{\perp} = 0$ by semiprimeness.

This suggests we look at the alternative analogue

(5.2) $W(z) = \{w \in A | (w\hat{A})z = w(\hat{A}z) = (z\hat{A})w = z(\hat{A}w) = 0\}$.

Since $zw = wz = 0$ and any associator $[z,A,w] = 0$, such w also satisfy $z(wA) = w(zA) = (Az)w = (Aw)z = 0$. Thus any product of z,w and one other factor vanishes. In particular $z \in W(z)^{\perp}$, and $z \in W(z)$ by strict triviality, so once more $z \in W(z) \cap W(z)^{\perp}$. We still have $B \cap B^{\perp} = 0$ for an ideal B in a semiprime alternative algebra, so if $z \neq 0$ the trouble must be that W(z) is not an ideal. How close is W(z) to being an ideal in the alternative case? It certainly is a linear subspace, and how far it is from being closed under multiplication is measured by how far each $Aw + wA$ is from satisfying the defining conditions (5.2) for W(z), namely how far $W(z,w) = \{(Aw+wA)\hat{A}\}z + (Aw+wA)\{\hat{A}z\} + \{z\hat{A}\}(Aw+wA) + z\{\hat{A}(Aw+wA)\}$ is from being zero. Thus if W(z) is not an ideal there must exist $w \in W(z)$ with $Aw+wA \not\subset W(z)$ and $W(z,w) \neq 0$. Note, however, that we cannot specify which is the offending w. Holding z and w fixed let us set

$$f(a,b) = [w,a,b]z, \qquad g(a,b) = z[a,b,w].$$

We claim $W = W(z,w)$ has the simple expression

(5.3) $$W = f(A,A) + g(A,A) \subset zA \cap Az .$$

This follows immediately from the following expressions for the terms constituting W:

$$f(a,b) = \{(wa)b\}z = \{b(aw)\}z = z\{(aw)b\} =$$
$$= U_{z,b}(aw) = (za)(wb) = (bw)(za) = (az)(bw)$$
$$(5.4) \qquad g(a,b) = z\{b(aw)\} = z\{(wa)b\} = \{b(wa)\}z$$
$$= U_{z,b}(wa) = (bw)(az) = (az)(wb) = (wb)(za)$$
$$f(a,b) + g(a,b) = \{(bw)a\}z = (wa)(bz) = z\{a(wb)\} = (zb)(aw).$$

The first of these follows from the following calculations, recalling that any product of z,w, and one other factor is zero:

$$[w,a,b]z = \{(wa)b\}z - \{w(ab)\}z$$
$$-[b,a,w]z = -\{(ba)w\}z + \{b(aw)\}z$$
$$U_{z,b}(aw) = z\{(aw)b\} + b\{(aw)z\} = \{z(aw)\}b + \{b(aw)\}z$$
$$= (za)(wb) + (ba)(wz) \quad \text{(Middle Moufang)}$$
$$U_{za,b}w = (za)\{wb\} + b\{w(za)\} = \{(za)w\}b + \{bw\}(za)$$
$$U_{a,bw}z = a\{z(bw)\} + (bw)\{za\} = \{az\}(bw) + \{(bw)z\}a.$$

The second follows by writing everything down backwards, i.e., applying the first to A^{op} (note $f(a,b)^{op} = \{(w\cdot a)\cdot b\}\cdot z = z\{b(wa)\} = -z[b,a,w] = z[a,b,w]$ $= g(a,b)$). The third follows from $f(a,b) + g(a,b) = \{b(aw+wa)\}z$ $= \{(ba)w\}z + \{(bw)a\}z$ and also $= \{(wa)b + b(wa)\}z = (wa)\{bz\} + b\{(wa)z\}$, then using duality.

We want to find the ideal generated by W. We begin by showing W is closed under commutators,

$$[A,W] \subset W.$$

By symmetry in (5.3) we need only show $[a,f(b,c)] \in W$ for all $a,b,c \in A$. We are claiming $af(b,c) \equiv f(b,c)a$ modulo W. But by (5.4) $af(b,c) = a \cdot U_{z,c}(bw)$ $= \{(az)(bw)\}c + \{(ac)(bw)\}z$ (Right Moufang) $= f(a,b)c + f(b,ac) \equiv f(a,b)c$ since $f(A,A) \subset W$. This shows $f(a,b)c$ is an alternating function of its arguments, congruent to zero if $a = b$ or $b = c$, so $af(b,c) \equiv f(a,b)c \equiv f(b,c)a$ as claimed.

This closure under commutators allows us to define $B = AW = WA$ without ambiguity. Note

$$(5.5) \qquad zB = 0$$

since $zB = z\{\hat{A}W\} \subset z\{\hat{A}(z\hat{A})\}$ (by (5.3)) $= (z\hat{A}z)A$ (by left Moufang) $= 0$ (by strict triviality). Now in a semiprime algebra $\hat{I}(z) \cap z^{\perp,L}$ contains no nonzero ideals (if $C \subset \hat{I}(z)$ has $zC = 0$ then $z \in Ann_L(C) \triangleleft A$ implies $C \subset \hat{I}(z) \subset Ann_L(C)$ so $CC = 0$ and C would be trivial). In our case $C = \hat{I}(W)$ is contained in $\hat{I}(z)$ since $W \subset \hat{I}(z)$, and $\hat{I}(W)$ is nonzero since W is nonzero, so $I(W)$ must not be annihilated by z: $\underline{zI(W) \neq 0}$.

Thus z kills B but not all of $\hat{I}(W) = \hat{I}(B)$. From the Commutator Derivation Formula III.2.10 we conclude $3[A,W,A] = [W,A \cdot A]-[W,A]A-A[W,A] \subset W-WA-AW \subset AW$. Therefore $3AB = 3A(AW) = 3(AA)W-3[A,A,W] \subset AW$, and similarly on the right, so

$$(5.6) \qquad 3AB + 3BA \subset B,$$

and B is close to being an ideal. If 3 is surjective then $3A = A$ and by (5.6) B is already an ideal; but then z kills $B = \hat{I}(B)$ by (5.5), which is impossible. If 3 is injective then z kills $3^{-\infty}B = \{x \in A \mid 3^n x \in B \text{ for some } n\}$ since $3^n zx = z(3^n x) \in zB = 0$ (by (5.5) implies $zx = 0$ by injectivity, and $3^{-\infty}B$ is an ideal by (5.6) since if $3^n x \in B$ then $3^{n+1}(xA+Ax) = 3\{(3^n x)A+A(3^n x)\}$ $\subset 3\{BA+AB\} \subset B$ and $xA+Ax \subset 3^{-\infty}B$; but then z kills $\hat{I}(W) = \hat{I}(B) \subset 3^{-\infty}B$, which is impossible.

Thus if 3 is injective or surjective, no $z \neq 0$ exists. ∎

In the associative case a trivial z generates a trivial $\hat{I}(z)$ by a trivial proof. In the alternative case the proof is not only highly nontrivial, it does not show $\hat{I}(z)$ is trivial — indeed it does not construct any specific trivial ideal out of z.

As with semiprimeness, we would like to be able to take an arbitrary algebra and remove an undesirable piece T(A) to obtain a strongly semiprime algebra A/T(A). The strongly semiprime radical T(A) of an alternative algebra is the smallest ideal $B \triangleleft A$ such that A/B is strongly semiprime. Such a B always exists: the intersection $B = \bigcap B_\alpha$ of all ideals B_α with strongly semiprime quotients A/B_α is certainly contained in all ideals with that property, yet it has the property itself since if $\bar{z} \in \bar{A} = A/B$ is trivial ($U_{\bar{z}}\bar{A} = \bar{0}$ or $U_z A \subset B$) then z is trivial mod all B_α ($U_z A \subset B \subset B_\alpha$), so belongs to all B_α by strong semiprimeness of A/B_α, hence z belongs to $\bigcap B_\alpha = B$ and $\bar{z} = \bar{0}$.

5.8 (Strong Semiprime Radical Theorem) Every alternative algebra A contains a unique smallest ideal T(A) such that A/T(A) is strongly semiprime. ∎

5.9 Corollary. A is strongly semiprime iff T(A) = 0. ∎

T(A) is the smallest ideal we can divide out by and still get rid of trivial elements. We would like a better idea of exactly what must be removed. Certainly we must get rid of the ideal Triv(A) generated by all trivial elements. In fact, as in 5.9

$$A \text{ is strongly semiprime} \iff \mathrm{Triv}(A) = 0.$$

WARNING: In general $T(A) > \mathrm{Triv}(A)$. Just because we have divided out by all trivial elements in A doesn't mean the result is trivia-free: there may well exist elements which are not themselves trivial, $U_z A \neq 0$, but become trivial in $\bar{A} = A/\mathrm{Triv}(A)$, $U_z A \subset \mathrm{Triv}(A)$.

In case there is trivia left over after dividing out by Triv(A), we must repeat the procedure. After a (possibly) transfinite number of steps we will reach T(A). (See Ex. 5.9).

This gives a fairly concrete picture of how to build T(A), but we would prefer some intrinsic information about T(A) and Triv(A). The first thing to note is that we can replace "generated by trivia" by "spanned by trivia".

5.11 **Lemma.** Triv(A) consists of all finite sums $z_1 + \cdots + z_n$ of trivial elements z_i.

Proof. Triv(A) certainly contains all such finite sums. Conversely these finite sums form a subset closed under addition (obviously!), scalar multiplication (since each αz_i is trivial if z_i is), and left or right multiplications (since each az_i, $z_i a$ is trivial if z_i is: both $U_{az} = L_a U_z R_a = 0$ and $U_{za} = R_a U_z L_a = 0$ if $U_z = 0$ by the Left and Right Fundamental Formulas). ∎

From this it is not hard (see Ex. 5.15) to derive one useful bit of information about Triv(A) — although it need not be nilpotent as an algebra, at least its elements are nilpotent: Triv(A) is a nil ideal.

We can actually do much better if we are willing to work at it: Triv(A) is locally nilpotent. The proof of this involves some combinatorial techniques which will come in handy later in studying polynomial identities (Appendix I). We begin with some preliminaries on Jordan polynomials.

Let Fralt$\{x_1,\cdots,x_n\}$ and Frass$\{x_1,\cdots,x_n\}$ be the free alternative and free associative algebras respectively on generators $x_1,\cdots,x_n$. We have a canonical parenthesis - deleting homomorphism Fralt $\overset{\sigma}{\to}$ Frass given by $x_i \to x_i$; we denote the image of $f = f(x_1,\cdots,x_n) \in$ Fralt by $f^\sigma = f^\sigma(x_1,\cdots,x_n) \in$ Frass. If a_i belong to an alternative algebra A we can evaluate f at the a_i: $f(a_1,\cdots,a_n)$. Similarly we can evaluate f^σ at ℓ_i in an associative algebra E: $f^\sigma(\ell_1,\cdots,\ell_n)$. A **Jordan polynomial** in Fralt is an element which can be built up from the generators x_i by taking linear combinations and Jordan products $(x^2, x\circ y, xyx)$.

The Left Moufang formula says the map $x \to L_x$ preserves linear combinations and Jordan products, so it also preserves Jordan polynomials.

5.12 (Generalized Left Moufang Formula) Left multiplication by a Jordan polynomial is the Jordan polynomial of left multiplications,

$$L_{p(a_1,\cdots,a_n)} = p^\sigma(L_{a_1},\cdots,L_{a_n})$$

whenever $p(x_1,\cdots,x_n)$ is a Jordan polynomial and $a_1,\cdots,a_n \in A$.

Proof. Since the Jordan polynomials in $x_1,\cdots,x_n$ are the elements of the smallest subspace of Fralt$(x_1,\cdots,x_n)$ containing the x_i and closed under Jordan products, it suffices to prove the set P = {alternative polynomials $p(x_1,\cdots,x_n) \mid L_{p(x_1,\cdots,x_n)} = p^\sigma(L_{x_1},\cdots,L_{x_n})\} = \{p \mid L_p = p^\sigma(L)\}$ is such a subspace.

Clearly P contains $p(x_1,\cdots,x_n) = x_i$ since $p^\sigma(x_n,\cdots,x_n) = x_i$ and $L_p = L_{x_i} = x_i(L_{x_i}) = p^\sigma(L_{x_i})$. It is linear since $L_{\alpha p+\beta q} = \alpha L_p+\beta L_q = \alpha p^\sigma(L)+\beta q^\sigma(L) = \{\alpha p+\beta q\}^\sigma(L)$ from the fact that $p \to p^\sigma$ is linear. Since $p \to p^\sigma$ is also a homomorphism of alternative algebras, $L_{pqp} = L_p L_q L_p$ (left Moufang) $= p^\sigma(L)q^\sigma(L)p^\sigma(L) = \{p^\sigma q^\sigma p^\sigma\}(L) = \{pqp\}^\sigma(L)$ and similarly $L_{p^2} = \{p^2\}^\sigma(L)$, so P is closed under pqp and p^2 and consequently under Jordan products. ∎

5.13 (Leading Term Theorem) If $w = x_n \cdots x_k$ is a word on an ordered alphabet $x_1 < \cdots < x_n$ which begins with x_n but ends with x_k for $k < n$, then there exists a Jordan monomial $p(x_1, \cdots, x_n)$ such that the associative polynomial $p^\sigma(x_1, \cdots, x_n)$ has w as its lexicographically leading constituent: $p^\sigma = w + \Sigma w_\alpha$ for $w_\alpha < w$ in the lexicographic order.

Proof. Note that a Jordan monomial need not be an alternative monomial: $x \circ y$ is a Jordan monomial but $xy+yx$ is an alternative polynomial rather than monomial.

Write $w = x_n^{e_1} w_1 \cdots x_n^{e_r} w_r$ for $w_i = w_i(x_1, \cdots, x_{n-1})$ words not involving x_n; by hypothesis $e_1 \geq 1$ and $\partial w_r \geq 1$. We will show there is p with $p^\sigma = w + \Sigma w_\alpha$ where w_α begins with x_i for $i < n$ (hence $w_\alpha < w$), and we do this by induction on r. If $r = 1$, $w = x_n^{e_1} x_{i_1} \cdots x_{i_s}$ $(i_k < n)$, then the only term beginning with x_n in p_1^σ for $p_1 = ((x_n^{e_1} \circ x_{i_1}) \circ \cdots) \circ x_{i_s}$ is $x_n^{e_1} x_{i_1} \cdots x_{i_s} = w$. If the only term beginning with x_n in p_{r-1}^σ is $x_n^{e_1} w_1 \cdots x_n^{e_{r-1}} w_{r-1}$ then the only term beginning with x_n in p_r^σ for $p_r = ((\{U_{p_{r-1}, x_{i_1}} x_n^{e_r}\} \circ x_{i_2}) \circ \cdots) \circ x_{i_s}$ is that of $p_{r-1} x_n^{e_r} x_{i_1} x_{i_2} \cdots x_{i_s}$, namely $x_n^{e_1} w_1 \cdots x_n^{e_{r-1}} w_{r-1} x_n^{e_r} x_{i_1} \cdots x_{i_s} = x_n^{e_1} w_1 \cdots x_n^{e_{r-1}} w_{r-1} x_n^{e_r} w_r = w$. ∎

5.14 Lemma. If $z_1, \cdots, z_n$ are trivial elements in an alternative algebra then any product $L_{z_{i_1}} \cdots L_{z_{i_N}}$ of left multiplications L_{z_i} of length $N = 2^n(n+1)!$ vanishes.

Proof. We induct on n, the case $n = 0$ being vacuous. Assume the result for n-1, so any monomial of length $N_0 = 2^{n-1} n!$ involving only $L_{z_1}, \cdots, L_{z_{n-1}}$ vanishes.

We can write any monomial of length $N = 2^n(n+1)!$ in $L_{z_1},\cdots,L_{z_n}$ as $w(L_{z_1},\cdots,L_{z_n}) = L_{z_{i_1}}\cdots L_{z_{i_N}}$ where $w(x_1,\cdots,x_n) = x_{i_1}\cdots x_{i_N}$ is an (associative) word on the alphabet of letters $x_1,\cdots,x_n$. We order this alphabet in the natural way $x_1 < x_2 <\cdots< x_n$ and prove the result by induction on the lexicographic order of the word w. This induction gets off the ground, since the lowest word of length N is x_1^N, and $L_{z_1}^N = L_{z_1^N} = 0$ (note $z_i^e = z_i(z_i^{e-2})z_i = 0$ if $e \geq 3$, and here $N \geq 2^1(1+1)! = 4$).

Assume the result $w'(L_{z_1},\cdots,L_{z_n}) = 0$ for lexicographically lower words $w' < w$. Write

$$w = w_o x_n^{e_1} w_1 \cdots w_{r-1} x_n^{e_r} w_r$$

for $e_i \geq 1$ and $w_1,\cdots,w_{r-1}$ nonempty words $w_i(x_1,\cdots,x_{n-1})$ not involving x_n. Here $w(L_{z_1},\cdots,L_{z_n})$ will vanish if any $e_i \geq 3$ (recall $L_{z_i}^e = L_{z_i^e} = 0$ if $e \geq 3$) or if any w_i has length $\geq N_o$ (by the induction hypothesis on n-1). Thus we may assume $e_i \leq 2$ and $\partial w_i < N_o$. Then $(n+1)2^n n! = N = \partial w = \sum_1^r e_i + \sum_o^r \partial w_i \leq \sum_1^r 2 + \sum_o^r N_o$ $= 2r+(r+1)N_o < (r+1)(2+N_o) \leq (r+1)2N_o = (r+1)2^n n!$ forces $n < r$.

If one of the w_i for $1 \leq i \leq r-1$ has degree 1, $w_i = x_j$, then $w(L_{z_1},\cdots,L_{z_n})$ $= 0$ since already $L_{z_n}^{e_i} L_{z_j} L_{z_n}^{e_{i+1}} = L_{z_n}^{e_i-1}\{L_{z_n z_j z_n}\}L_{z_n}^{e_{i+1}-1} = 0$ by $z_n A z_n = 0$. Thus we may assume the monomials $w_1,\cdots,w_{r-1}$ which are surrounded by x_n's have degree ≥ 2; there are $r-1 > n-1$ of these and only $n-1$ variables $x_1,\cdots,x_{n-1}$, so two end in the same x_k: $w_i = w_i'x_k$, $w_j = w_j'x_k$ for $1 \leq i < j \leq r-1$. By 5.13 there is a Jordan monomial $p(x_1,\cdots,x_n)$ having $v = x_n^{e_{i+1}} w_{i+1} \cdots x_n^{e_j} w_j'$ as lexicographically leading monomial,

$$p(x_1,\cdots,x_n) = v + \Sigma v_\alpha$$

for monomials (words) v_α of the same degree but lexicographically lower than v. Here v is a middle segment of w, $w = w'vw''$ for $w' = w_o x_n^{e_1}\cdots x_n^{e_i} w_i$ $= (w_o\cdots w_i')x_k = u'x_k$ and $w'' = x_k x_n^{e_{j+1}} w_{j+1}\cdots x_n^{e_r} w_r = x_k u''$. Because the v_α are lexicographically lower than v (but of the same degree) the $w_\alpha = w'v_\alpha w''$ are lecicographically lower than $w = w'vw''$ (but of the same degree N): since they both begin with w', the first place $w'v_\alpha w''$ and $w'vw''$ differ is the first place where v_α differs from v, and in that place v_α has the lower letter, so $w'v_\alpha w''$ has a lower letter than $w'vw''$ in the first place they differ. Thus by lexicographic induction $w_\alpha(L_{z_1},\cdots,L_{z_n}) = 0$. From

$$w = w'vw'' = w'(p-\Sigma v_\alpha)w'' = w'pw'' - \Sigma w_\alpha$$

we see

$$\begin{aligned} w(L_{z_1},\cdots,L_{z_n}) &= w'(L_{z_1},\cdots,L_{z_n})p(L_{z_1},\cdots,L_{z_n})w''(L_{z_1},\cdots,L_{z_n}) \\ &= u'(L_{z_1},\cdots,L_{z_n})L_{z_k}L_{p(z_1,\cdots,z_n)}L_{z_k}u''(L_{z_1},\cdots,L_{z_n}) \quad \text{(by 5.12)} \\ &= u'(L_{z_1},\cdots,L_{z_n})L_{z_k p(z_1,\cdots,z_n) z_k}u''(L_{z_1},\cdots,L_{z_n}) \quad \text{(left Moufang)} \\ &= 0. \end{aligned}$$

from $z_k A z_k = 0$. This completes the lexicographic subinduction on w and the degree induction on n. $\blacksquare$

5.13 (Slinko's Local Theorem) An alternative algebra generated by a finite number of trivial elements is nilpotent.

Proof. Let A be generated by trivial elements $z_1,\cdots,z_n$. By the Normal Form Theorem for Elements I.7.10, every element of A^k is a linear combination of 2nd order monomials $w_1(w_2(\cdots w_r))$ where each $w_i = z_{i_1}(z_{i_2}((\cdots z_{i_s}))$ is a 1st order monomial in the generators z_j, and where the degrees of the w_i add up to at least k: $\partial w_1 + \cdots + \partial w_r \geq k$.

By Lemma 5.14 we know there is an integer $N = N(t)$ such that $L_{y_{i_1}} \cdots L_{y_{i_{d-1}}} = 0$ whenever $y_1, \cdots, y_t$ are trivial and $d \geq N$. Thus all our $w_i = L_{z_{i_1}} L_{z_{i_2}} \cdots L_{z_{i_{s-1}}} (z_{i_s})$ of degree $s = \partial w_i \geq N(n)$ are zero. Consider the finite number $w_1, \cdots, w_m$ of nonzero 1st order monomials w_i of degree $< N(n)$. Once more $w_{i_1}(w_{i_2}(\cdots w_{i_r})) = L_{w_{i_1}} L_{w_{i_2}} \cdots L_{w_{i_{r-1}}}(w_{i_r}) = 0$ for $r \geq N(m)$. But then $A^{N(n)N(m)}$ is spanned by $w_1(w_2(\cdots w_r))$ for $\partial w_1 + \cdots + \partial w_r \geq N(n)N(m)$, where this monomial vanishes if some $\partial w_i \geq N(n)$, and if all $\partial w_i < N(n)$ then $rN(n) > \partial w_1 + \cdots + \partial w_r \geq N(m)N(n)$ implies $r \geq N(m)$ so $w_1(w_2(\cdots w_r)) = 0$ anyway. Thus $A^{N(n)N(m)} = 0$ and A is nilpotent. ∎

A global version of the theorem is

5.16 (Slinko's Global Theorem) An alternative algebra which is generated by trivial elements is locally nilpotent.

Proof. Let B be a finitely generated subalgebra of an algebra A generated by trivial elements; we must prove B is nilpotent. Now each of the finitely many generators $b_1, \cdots, b_n$ of B is a polynomial $b_i = p_i(z_{i1}, \cdots, z_{in(i)})$ in a finite number of trivial z_{ij} by our hypothesis that A is generated by trivial elements so B is contained in the subalgebra C generated by the finite number of trivial elements z_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq n(i))$. By the local version of the theorem C is nilpotent, hence its subalgebra B is nilpotent too. ∎

5.17 Corollary. If A is an alternative algebra then Triv(A) is locally nilpotent: Triv(A) $\subset$ L(A). ∎

5.18 Corollary. If A contains no locally nilpotent ideals, it is strongly semiprime. ∎

5.19 Corollary. T(A) $\subset$ L(A). ∎

Exercises IV.5

5.1 It is not in general true that if $\alpha \in \Phi$ is surjective on A then A can be imbedded in an algebra $\tilde{A}$ on which α is bijective (since A itself can have α-torsion); however, it is true when A is semiprime. Show $A[\gamma] = \{a \in A \mid \gamma a = 0\}$ is a γ-invariant ideal, as is $A[\gamma^\infty] = \{a \mid$ some $\gamma^k a = 0\}$, and $\bar{A} = A/A[\gamma^\infty]$ has no γ-torsion ($\gamma \in \Gamma(A)$ in the centroid). If A is semiprime show $\gamma^n x = 0 \Rightarrow \gamma x = 0$, $A[\gamma^\infty] = A[\gamma]$. Show that if α is surjective on a semiprime A, it is bijective. If γ,δ are relatively prime ($1 = \alpha\gamma+\beta\delta$) show $A[\gamma] \cap A[\delta] = 0$.

5.2 If A has no 3-torsion show A is imbedded in $A_{(3)} = A \otimes Q_{(3)}$ ($Q_{(3)}$ = rationals with denominator a power of 3). Show 3 is bijective on $A_{(3)}$, and $A_{(3)}$ is semiprime iff A is.

Exercises 5.1 and 5.2 show we could have assumed $\frac{1}{3} \in \Phi$ without loss of generality in proving the Strong Semiprimeness Theorem.

5.3 Finish the proof of the Strong Semiprimeness Theorem as follows: using (5.5) and (5.6) show $3VB = 0$ and $(3B)^2 = 0$. Use Semiprime Inheritance to conclude $3VA = 0$. If 3 is injective or surjective conclude $V = 0$.

5.4 Prove the <u>Strong Semiprimeness Inheritance Theorem</u>. If A is strongly semiprime, so is any ideal $B \triangleleft A$ or any Peirce subalgebra eAe ($e \in A$ idempotent).

5.5 Show by example that A strongly semiprime does <u>not</u> imply all homomorphic images $F(A) \cong A/B$ are strongly semiprime.

5.6 Prove that strong semiprimeness is recoverable: if A/B and B are strongly semiprime, so is A.

5.7 Prove that a subdirect sum of strongly semiprime algebras is again strongly semiprime. Conclude that a direct sum or product inherits strong semiprimeness from the factors.

5.8 Show directly from the definition that $T(A) = A$ iff all nonzero homomorphic images of A contain trivial elements, and $T(A) = 0$ iff A has no trivial elements.

5.9 Establish the Recursive Construction: If we define $T_\lambda(A)$ for ordinals λ recursively by $T_o(A) = 0$, $T_{\lambda+1}(A) = Triv(A, T_\lambda(A)) =$ {ideal generated by all elements trivial mod $T_\lambda(A)$}, and $T_\lambda(A) = \bigcup_{\mu<\lambda} T_\mu(A)$ for a limit ordinal λ, then $T_\lambda(A) = T(A)$ for $||\lambda|| > |A|$.

5.10 If B is an ideal in A, show $B/B \cap T(A)$ is strongly semiprime; conclude $T(A) \subset B \cap T(A)$. Use the recursive construction of Ex. 5.9 to show $B \cap T_\lambda(A) \subset T(B)$ at each stage. Deduce the Strong Semiprime Radical Inheritance Theorem: If B is an ideal in A then

$$T(B) = B \cap T(A).$$

5.11 Prove Ex. 5.10 by showing $B \cap T_\lambda(A) = T_\lambda(B)$ at each inductive step.

5.12 An element z of an arbitrary nonassociative algebra A is trivial if $z(Az) = 0$. Show there exists a smallest ideal $T(A) \triangleleft A$ such that $A/T(A)$ has no trivial elements. Show $z(Az) \subset T(A) \Rightarrow z \in T(A)$.

5.13 A u-sequence is a sequence $x_0, x_1, \cdots$ where $x_{n+1} = x_n(y_n x_n)$ for some y_n; the sequence begins with x_0 and terminates if one $x_n = 0$ (hence all $x_m = 0$ for $m \geq n$). Show if $x \notin T(A)$ there exists a non-terminating u-sequence beginning with x_0. If B is maximal among all ideals missing a non-terminating u-sequence show B is prime in A, i.e., $CD \subset B$ imply one of the ideals C or D is contained in B.

5.14 In an arbitrary nonassociative algebra A, show

$$T(A) \supset \{x \mid \text{all u-sequences beginning with x terminate}\}$$
$$\supset \bigcap \{\text{prime ideals}\} = S(A).$$

5.15 Show that if y is nilpotent and z trivial then x = y+z is nilpotent.

Conclude that Triv (A) is a nil ideal.