§4. Local nilpotence

Although solvability and nilpotence are not equivalent in general, they coincide in the presence of certain finiteness conditions. We have already seen this for algebras with a.c.c. or d.c.c.; in the present section we establish Zhevlakov's Equivalence Theorem that for finitely generated algebras solvability, nilpotence, and nilpotence of all multiplication algebras are equivalent. This shows that local solvability and local nilpotence are equivalent, and allows us to define the locally nilpotent or Levitzki radical.

Finitely generated algebras

The methods to be used in the finitely generated case have a combinatorial flavor. We begin by obtaining a "monotone normal form" for multiplication operators, where we linearly order the monomials in $x_1,\cdots,x_n$ in such a way as to respect degree: we linearly order the monomials $p = p(x_1,\cdots,x_n)$ of a given degree in some (arbitrary but fixed) fashion, and define $p < q$ for monomials of different degrees $\deg p < \deg q$.

4.1 (Monotone Normal Form Theorem) Every multiplication operator of degree m in the variables $x_1,\cdots,x_n$ can be written as a linear combination of monomial operators

$$M = L_{y_1}\cdots L_{y_r}R_{y_0} \quad (y_1 < \cdots < y_r)$$

of degree m for suitable monomials $y_1,\cdots,y_r \in A$ and $y_0 \in \hat{A}$.

Proof. By the Left Normal Form Theorem I.7.9 we know any multiplication of degree m can be written as a linear combination of operators $L_{y_1}\cdots L_{y_r}R_{y_0}$ of degree m, where the y_i are first-order monomials in the x_i in left-normal form but are not necessarily increasing. When we start to straighten them so $y_1 < \cdots < y_r$ we will introduce some left-abnormal monomials.

For given degree m we show $L_{y_1} \cdots L_{y_r} R_{y_0}$ can be written as a linear combination of $L_{z_1} \cdots L_{z_s} R_{z_0}$ of degree m with $z_1 < \cdots < z_0$ by induction on the length r. If $r = 1$ then $L_{y_1} R_{y_0}$ is already straight. Assume now all $L_{y_1} \cdots L_{y_s} R_{y_0}$ of degree m but length $s < r$ can be straightened, and consider $L_{y_1} \cdots L_{y_r} R_{y_0}$ of length r. If at some point $y_i > y_{i+1}$ then $L_{y_i} L_{y_{i+1}} = L_{y_{i+1}} L_{y_i} + L_{y_i y_{i+1}} + L_{y_{i+1} y_i}$ shows we can interchange L_{y_i} and $L_{y_{i+1}}$ modulo terms of the same degree but shorter length r-1 (which we can straighten by the induction hypothesis on r). Repeating this we can arrange it so $y_1 \leq \cdots \leq y_r$. If at some point $y_i = y_{i+1}$ then $L_{y_i} L_{y_{i+1}} = L_{y_i} L_{y_i} = L_{y_i^2}$ can again be replaced by a shorter term. Thus the given $L_{y_1} \cdots L_{y_r} R_{y_0}$ can be expressed as a sum of terms $L_{z_1} \cdots L_{z_r} R_{z_0}$ with $z_1 < \cdots < z_r$ together with shorter (hence inductively straightenable) terms. ∎

Using this normalization we construct certain constants. Recall from I.7 the definition of the multiplication ideal M(B;A) generated by M(B) in M(A).

4.2 (Zhevlakov's Constants) There exist universal constants $Z(n,m)$, $W(n,m)$ such that whenever A is an alternative algebra with n generators we have

(i) $A^{Z(n,m)} \subset D^m(A)$

(ii) $M(A)^{W(n,m)} \subset M(D^{m-1}(A);A)$

Proof. We fix n and induct on m. The result for $m = 1$ is trivial: $Z(n,1) = 2$ has $A^2 = D(A)$, $W(n,1) = 1$ has $M(A) = M(D^0(A);A)$. Assume we have found $Z(n,m-1)$ and $W(n,m-1)$ satisfying (i) and (ii). We will use $Z(n,m-1)$ to construct $W(n,m)$ (this explains why we have m-1 instead of m in (ii)), then use $W(n,m)$ to construct $Z(n,m)$.

Set

$$W(n,m) = \{N(n,Z(n,m-1))+1\}Z(n,m-1)$$

where in general $N(n,d)$ denotes the number of (nonassociative) monomials of degree $<d$ in n variables. By the Monotone Normal Form Theorem 4.1, the span $M(A)^{W(n,m)}$ of operators of degree $\geq W(n,m)$ can be spanned by monomial operators

$$M = L_{y_1} \cdots L_{y_r} R_{y_0}$$

of degree $\geq W(n,m)$ for suitable distinct monomials $y_1 < \cdots < y_r$ in the generators $x_1, \cdots, x_n$ for A. If all the y_i were to have degree $< Z(n,m-1)$ then (since the y_i are distinct, and there are at most $N(n,Z(n,m-1))$ such distinct monomials of degree $<Z(n,m-1)$) we must have $r \leq N(n,Z(n,m-1))$ in which case M would have total degree

$$\deg M = \deg y_1 + \cdots + \deg y_r + \deg y_0 < (r+1)Z(n,m-1)$$

$$\leq \{N(n,Z(n,m-1))+1\}Z(n,m-1) = W(n,m)$$

whereas it actually has degree $\geq W(n,m)$. Thus when we apportion the generators among the $y_1, \cdots, y_r, y_0$ at least one of these y_i must receive a portion of $\geq Z(n,m-1)$ generators by the Pigeonhole Principle. By the induction hypothesis (i) on $Z(n,m-1)$, such a y_i of degree $\geq Z(n,m-1)$ falls in $D^{m-1}(A)$, so the multiplication operator M involving y_i falls in the multiplication ideal $M(D^{m-1}(A);A)$ as required by (ii).

Once (ii) has been verified, set

$$Z(n,m) = W(n,m) + Z(n,m-1)$$

Then $A^{Z(n,m)} = M(A)^{W(n,m)} A^{Z(n,m-1)}$ (by (1.8)) $\subseteq M(D^{m-1}(A);A)D^{m-1}(A)$ (by (ii) and induction (i)) $\subset D^{m-1}(A)^2$ (note that if we hit $D^{m-1}(A)$ by a multiplication from A we stay inside the ideal $D^{m-1}(A)$, hitting it with $D^{m-1}(A)$ moves it into $D^{m-1}(A)^2$, and this ideal remains impervious to further hitting with A) $= D^m(A)$ as required by (i). This completes the inductive construction of Z and W. ■

4.3 Remark. The basic idea of Zhevlakov's proof is that a multiplication operator of high total degree involving distinct monomials must involve some monomials of high degree, for there are only a small number of different monomials of low degree.

Thus the idea is to build up the L_y until at least one has a high degree. This is just the opposite of the procedure used in the Generation Lemma I.7.6, where we tried to tear down L_y's into pieces L_{x_i} of degree one. ■

By their form these Z(n,m) and W(n,m) are universal: they depend only on the N(n,d) and the previous Z's and W's. Another way to see their universality is to observe that if we have found particular Z's and W's for the free alternative algebra A on n generators, the same Z's and W's will work for any algebra $\tilde{A}$ with n generators: $\tilde{A}$ is a homomorphic image of A, so simply apply the homomorphism to (i) and (ii) for A to obtain the corresponding relations for $\tilde{A}$.

This is a general phenomenon: particular constants for the free algebra are universal because the free algebra is universal.

This combinatorial result leads directly to

4.4 (Zhevlakov's Equivalence Theorem) The following are equivalent for a finitely generated algernative algebra:

(i) A is solvable

(ii) A is nilpotent

(iii) M(A) is nilpotent

(iv) A acts nilpotently on all bimodules

(v) all multiplication algebras $M_E(A|M)$ are nilpotent.

Proof. To say A acts nilpotently on M means $M_{E_0}(A|M)$ is nilpotent for $E_0 = A \oplus M$ the split null extension. But in $M_E(A|M)$ for an arbitrary enveloping algebra E, only the action of A on M is relevant, so we get the same multiplication algebra

$$M_E(A|M) = M_{E_0}(A|M)$$

from the split null extension E_0 that we do from E. Thus we see (v) $\Leftrightarrow$ (iv). When M is the regular bimodule, $M_E(A|M)$ reduces to the ordinary multiplication algebra M(A), so (iv) $\Rightarrow$ (iii). By Etherington's Theorem (iii) $\Leftrightarrow$ (ii), and always (ii) $\Rightarrow$ (i).

To show (i) $\Rightarrow$ (iv) and close the cycle of implications, we will prove that if A has n generators and is solvable of index d then

$$M_{E_0}(A|M)^{k-1} = 0 \qquad (k = Z(n+1,d+1)).$$

To show these operators are zero it suffices to show they kill everything they act on, $M_{E_0}(A|M)^{k-1}z = 0$ for all $z \in M$. If B denotes the subalgebra of E_0 generated by A and z, it suffices if $B^k = 0$ (since $M_{E_0}(A|M)^{k-1}z \subset M(B)^{k-1}B \subset B^k$). But B has n+1 generators (namely z together with the n generators of A), and is solvable of index d+1 (since $D^d(A) = 0$ implies $D^d(B) \subset D^d(E_0) \subset D^d(A)+M = M$ and therefore $D^{d+1}(B) \subset D(M) = 0$; here it is crucial that E_0 is the split null extension, for in a general enveloping algebra E the space M need not be trivial), so $B^k = B^{Z(n+1,d+1)} \subset D^{d+1}(B) = 0$ by choice of k as a Zhevlakov constant. ∎

Algebras which are finitely spanned are finitely generated, so

4.5 Corollary. Solvability and nilpotence coincide for algebras which are finitely spanned (eg. finite-dimensional over a field). ∎

Local nilpotence

A nonassociative algebra is **locally nilpotent** if all its finitely generated subalgebras are nilpotent. Similarly we can define **local solvability** . Since a finitely generated subalgebra of an alternative algebra is nilpotent iff it is solvable, <u>local solvability and local nilpotence coincide in the alternative case.</u>

Our previous equivalence concerning nilpotence in a finitely generated algebra translates into an equivalence concerning local nilpotence in an arbitrary algebra.

4.6 (Equivalence Theorem for Local Nilpotence) The following are equivalent for an alternative algebra A:

(i) A is locally nilpotent

(ii) A is locally solvable

(iii) M(A) is a locally nilpotent algebra of transformations

(iv) A acts locally nilpotently on any bimodule M.

(v) all multiplication algebras $M_E(A|M)$ are locally nilpotent.

Proof. We already observed (i) $\Leftrightarrow$ (ii) and (v) $\Leftrightarrow$ (iv) since any $M_E(A|M)$ coincides with $M_{E_0}(A|M)$ for the split null extension E_0; (iv) $\Rightarrow$ (iii) by taking the regular bimodule $M = A$, (iii) $\Rightarrow$ (i) since if A_0 is a finitely generated subalgebra of A the multiplication algebra $M_A(A_0)$ is a finitely generated subalgebra of $M_A(A)$ (by the Generation Lemma), therefore is nilpotent if $M_A(A)$ is locally nilpotent; but then its restriction $M(A_0) = M_A(A_0)|_{A_0}$ to the invariant subspace A_0 is also nilpotent, so that A_0 is nilpotent by Etherington.

Finally, we show (i) $\Rightarrow$ (v). Let M_0 be a finitely generated subalgebra of $M_E(A|M)$: the finite number of generators of M_0 involve only a finite number of elements $x_1,\cdots,x_n$ from A, so $A_0 = \Phi[x_1,\cdots,x_n]$ is a finitely generated subalgebra of A and M_0 is contained in the subalgebra $M_E(A_0|M)$ of $M_E(A|M)$ generated by all ℓ_x, r_x for $x \in A_0$. If A is locally nilpotent, A_0 is nilpotent and so (by Zhevlakov Equivalence) the multiplication algebra $M_E(A_0|M)$ is nilpotent too, as is its subalgebra M_0. Thus every finitely generated subalgebra M_0 of $M_E(A|M)$ is nilpotent. ∎

In dealing with local solvability, it is important that the derived algebras of a finitely generated algebra remain finitely generated.

4.7 Lemma. If an alternative algebra A is finitely generated, so is any derived $D^n(A)$.

Proof. It suffices to prove D(A) remains finitely generated, since we can then repeat the procedure n times. If A has n generators $a_1,\cdots,a_n$ we claim D(A) is generated by all $p(a_1,\cdots,a_n)$ where $p(x_1,\cdots,x_n)$ is a free monomial of degree $2 \le \deg p < Z(n,2)$.

Since $D(A) = A^2$ is spanned by monomials $q(a_1,\cdots,a_n)$ of degree ≥ 2, it suffices if each such q is generated by the p's. We induct on the degree of q. If $2 \leq \deg q < Z(n,2)$ then trivially q is generated by p's, since it is one of the p's! On the other hand, if $\deg q \geq Z(n,2)$ then by definition of the Zhevlakov constants (in the free algebra $B = \mathrm{Fralt}[x_1,\cdots,x_n]$) $q(x_1,\cdots,x_n) \in B^{Z(n,2)}$ implies $q(x_1,\cdots,x_n) = \Sigma q_i'(x_1,\cdots,x_n)q_i''(x_1,\cdots,x_n) \in D^2(B) = (B^2)^2$ for monomials $q_i', q_i'' \in B^2$ of degree ≥ 2; BY DEGREE CONSIDERATIONS IN THE FREE ALGEBRA we can assume $\deg q_i' + \deg q_i'' = \deg q$, so $2 \leq \deg q_i', q_i'' < \deg q$. By induction these lower-degree $q_i'(a_1,\cdots,a_n)$, $q_i''(a_1,\cdots,a_n)$ are generated by the $p(a_1,\cdots,a_n)$'s, so $q(a_1,\cdots,a_n) = \Sigma q_i'(a_1,\cdots,a_n)q_i''(a_1,\cdots,a_n)$ is too. ∎

The fact that local nilpotence coincides with local solvability makes it a radical property.

4.8 (Radical Property for Local Nilpotence) Local nilpotence is a very strongly hereditary radical property for alternative algebras: (i) if an alternative algebra A is locally nilpotent so is any subalgebra and every homomorphic image, (ii) if B and A/B are locally nilpotent, so is A, (iii) the union $B = \bigcup B_\alpha$ of a chain of locally nilpotent ideals is locally nilpotent. Therefore each alternative algebra A contains a largest locally nilpotent ideal L(A), which is also the smallest ideal whose quotient has no locally nilpotent ideals.

Proof. (i) If B is a subalgebra of a locally nilpotent A, any finitely generated subalgebra B_0 of B is a finitely generated subalgebra of A and consequently nilpotent. This shows B is locally nilpotent.

Any finitely generated subalgebra $\tilde{A}_0$ of a homomorphic image $\tilde{A} = F(A)$ is the image of $\tilde{A}_0 = F(\tilde{A}_0)$ of a finitely generated subalgebra A_0 (pick preimages of the generators of $\tilde{A}_0$), and since A_0 is nilpotent by hypothesis on A so is its image $\tilde{A}_0$. Thus $\tilde{A}$ is locally nilpotent.

(ii) Suppose B and $\bar{A} = A/B$ are locally nilpotent. If A_0 is a finitely generated subalgebra of A then $\bar{A}_0$ is a finitely generated subalgebra of $\bar{A}$, hence nilpotent by local nilpotence of $\bar{A}$. In particular it is solvable, say $D^n(\bar{A}_0) = \bar{0}$ or $D^n(A_0) = B_0 \subset B$. Now by the Lemma B_0 remains finitely generated (which is one reason we used $D^n(A_0)$ instead of A_0^n), so by local nilpotence of B we have $D^m(B_0) = 0$. Therefore $D^{n+m}(A_0) = D^m(D^n(A_0)) = D^m(B_0) = 0$ (this is the other reason we deal with derived algebras rather than powers), and A_0 is at least solvable. Because it is also finitely generated, it is nilpotent by Zhevlakov Equivalence. Thus all finitely generated A_0 are nilpotent, and A is locally nilpotent.

(iii) Any finitely generated subalgebra B_0 of $B = \bigcup B_\alpha$ is contained in some B_α when α is large enough to contain the generators of B_0, therefore is nilpotent if B_α is locally nilpotent. This shows B itself is locally nilpotent.

From (i)-(iii) it follows as usual that the sum or union of all locally nilpotent ideals is the unique maximal locally nilpotent ideal L(A), and A/L(A) contains no locally nilpotent ideals. ■

Thus there always exists a maximal locally nilpotent ideal in A, the **locally nilpotent** or **Levitzi radical**

$$L(A).$$

We have $L(A/L(A)) = 0$, so that A/L(A) contains no locally nilpotent ideals.

Since a (globally) nilpotent ideal is most certainly also locally nilpotent, A/L(A) is semiprime. By minimality of the semiprime radical we have

$$S(A) \subset L(A)$$

and these coincide if L(A) is finitely generated (eg. if A is finite-dimensional over a field).

Simple algebras could conceivably be nil, but they can't be nilpotent or antiprime (by 3.2); it is important that they can't even be locally nilpotent. For arbitrary nonassociative algebras we have

4.9 (Zhevlakov-Slater Lemma) If $z \in M(A)z$ where A is a locally nilpotent linear algebra, then $z = 0$.

Proof. If $z = Mz$ then $z = Mz = M^2z = \cdots = M^nz$. If A_0 is the subalgebra generated by z together with the finite number of elements in the multiplication operator M, then by local nilpotence eventually $z = M^nz \in A_0^{n+1} = 0$. ∎

4.10 (Nonsimplicity Theorem for Local Nilpotence) A simple nonassociative algebra cannot be locally nilpotent.

Proof. If A is simple we have $A^2 \neq 0$ by definition, so $A^{\perp} \neq A$; since $A^{\perp}$ is always an ideal, by simplicity the only other possibility is $A^{\perp} = 0$. Thus $z \neq 0 \Rightarrow z \notin A^{\perp} \Rightarrow M(A)z \neq 0$; since $M(A)z$ is an ideal, by simplicity it must be all of A. Thus $z \in M(A)z$ for all $z \neq 0$, so by the Zhevlakov-Slater Lemma A is not locally nilpotent. ∎

We have seen that $L(A) = S(A)$ for finitely generated algebras; the same is true for algebras with d.c.c. on ideals.

4.11 (Zhevlakov Nilpotence Theorem) If A has d.c.c. on ideals then the Levitzki radical $L(A)$ is nilpotent. In particular, $L(A) = S(A)$ is the maximal nilpotent ideal, and a semiprime algebra with d.c.c. contains no locally nilpotent ideals.

Proof. Always $S(A) \subset L(A)$, and to prove inclusion in the other direction it suffices if $\overline{L(A)} = 0$ in $\bar{A} = A/S(A)$, ie. show a semiprime algebra with d.c.c. has no locally nilpotent ideals. But if locally nilpotent ideals exist we can choose a minimal one; since such an ideal can't be a simple algebra by 4.10, according to the Minimal Ideal Theorem IV.1.11 it must be trivial, contrary to semiprimeness.

Thus $L(A) = S(A)$. We know by the Zhevlakov-Slater Nilpotence Theorem 2.16 that $S(A)$ is the maximal nilpotent ideal in the presence of the d.c.c. ∎

Exercises

4.1 Prove the Theorem: A minimum ideal B of a locally nilpotent algebra A is trivial, indeed $M(A)B = 0$.

4.2 Prove the Finite Generation Theorem: There are integers $d(n,k)$ such that whenever an alternative algebra A is generated by n elements its derived algebras $D^k(A)$ are generated by $d(n,k)$ elements.

4.3 Use the Zhevlakov-Slater Lemma and Nilpotence Theorem to show that if A is locally nilpotent with d.c.c. on ideals then A is nilpotent (avoid the Minimal Ideal Theorem!)

4.4 This does not show $L(A)$ is nilpotent when A has d.c.c. (we would need d.c.c. on all $L(A)$-ideals, not just on all A-ideals inside $L(A)$). However, show it does imply $L(A)$ is nilpotent when A has d.c.c. on inner ideals.

4.5 An alternative proof avoids Zhevlakov-Slater Nilpotence. Show that if A is any nonassociative algebra over a field Φ with d.c.c. on ideals, then A/A^2 is finite dimensional. Conclude there is a finitely generated subalgebra B with $A = B+A^m$ for all m. If A is alternative and B is nilpotent, use Zhevlakov's Equivalence Theorem to show $M(A)^n = M(A)^{n+1} = \cdots$ for some n. Using d.c.c. and the Zhevlakov-Slater Lemma, conclude $M(A)^n A = 0$, therefore establishing the weaker Theorem: If A is a locally nilpotent algebra over a field Φ with d.c.c. on ideals, then A is nilpotent.

4.6 If B,C are ideals in a nonassociative algebra show $I(B,C) = \{T \in M(A) \mid T(B) \subset C\}$ is an ideal in $M(A)$. If A is such that squares of ideals are ideals, show $L_y, R_y \in I(D(A), D^2(A))$ for $y \in D(A)$. Conclude $f(y_1,\cdots,y_r) = M_{y_1}\cdots M_{y_r}$ (for fixed choice of each M as an L or R) is an alternating function modulo $I(D(A),D^2(A))$ of its variables $y_1,\cdots,y_r \in A$ vanishing on $D(A)$. If A is alternative with n generators show $M(A)^{n+1} \subset I(D(A),D^2(A))$; deduce $A^{n+3} \subset D^2(A)$.

IV.4.1 Problem Set on Dorofeev Equivalence

The initial part of the discussion of the equivalence between solvability and nilpotence applies to a more general class of algebras than just the alternative algebras. We have seen that the k^{th} power A^k is always an ideal in any linear algebra A. However, the k^{th} derived algebras $D^k(A)$ are not always ideals in A; we restrict our attention to algebras where they are,

(Ideal Axiom) $\quad D^k(A) \triangleleft A \quad$ for all k.

We have seen from the Product Theorem that alternative algebras satisfy the Ideal Axiom.

Consider the following 5 conditions on a variety of algebras satisfying the Ideal Axiom:

(i) a finitely generated algebra of the variety is solvable iff it is nilpotent

(ii) a finitely generated algebra of the variety which is solvable of index 2 is nilpotent

(iii) for each finitely generated algebra of the variety there is an an integer $d = d(A)$ that $A^d \subset D^2(A)$

(iv) for each finitely generated algebra of the variety there are integers $d(k) = d(k,A)$ such that $A^{d(k)} \subset D^k(A)$

(v) an algebra of the variety is locally solvable iff it is locally nilpotent.

1. Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

2. Use the existence of free algebras in a variety and the notion of degree to prove that if an algebra A in the variety is finitely generated, so is its square A^2.

3. If for each finitely generated algebra B in the variety there is an integer $d(k,B)$ with $B^{d(k,B)} \subset D^k(B)$ for some fixed k (and also for $k = 2$), show $A^{ig} \subset D(A)^i + D^{k+1}(A)$ for all i and all finitely generated A in the variety ($g = \max\{d(2,A), d(k,A), d(k,A^2)\}$).
4. Conclude $A^{d(k+1,A)} \subset D^{k+1}(A)$ for $d(k+1,A) = g^2$.
5. Deduce the Dorofeev Equivalence Theorem. In a variety of algebras satisfying the Ideal Axiom, conditions (i)-(v) are equivalent.
6. Show the Dorofeev constants $d(k,A)$ depend only on k and the number of generators of A, not on A itself.
7. Use Dorofeev Equivalence and Exercise 4.6 to show local nilpotence equals local solvability in alternative algebras.

IV.4.2 Problem Set: Alternate Proof of Zhevlakov Nilpotence

1. If A is an arbitrary nonassociative algebra with d.c.c. which is not nilpotent, show there is a minimal ideal B on which it is not nilpotent. Show $M(A)^k B = B$ for all k, yet if A is locally nilpotent for each $b \in B$ there is an $n = n(b)$ with $M(A)^n b = 0$. We would like to be able to choose a universal n that works for all b, thus reaching a contradiction.
2. To pass from the given Φ to a field $\tilde{\Phi}$, show there is an ideal $\Omega \triangleleft \Phi$ with $\omega B = 0$ for $\omega \in \Omega$ and $\lambda B = B$ for $\lambda \notin \Omega$. Conclude B is a module over a integral domain $\bar{\Phi} = \Phi/\Omega$. Show its torsion part $B_0 = \{b \mid \lambda b = 0$ for some $\lambda \notin \Omega\}$ is an ideal with $M(A)B_0 = 0$, using $M(A_\lambda)B = B$ for $A_\lambda = \bigcap \lambda^n A$.
3. Show that if $\bar{M}$ is a torsion-free module over an integral domain $\bar{\Phi}$, there is a lattice isomorphicm between the $\bar{\Phi}$-submodules of $\bar{M}$ and the $\tilde{\Phi}$-submodules of the module of fractions $\tilde{M} = \tilde{\Phi} \otimes_{\bar{\Phi}} \bar{M}$ ($\tilde{\Phi}$ the field of fractions of $\bar{\Phi}$). If $\bar{M}$ has d.c.c. on submodules, conclude it also has the a.c.c.; in particular, it is finitely spanned over $\bar{\Phi}$.
4. When A is alternative, show $K = \{a \in A \mid$ for all $b \in B$, ab and ba lie in $\bar{\Phi}^{-1} \cdot M(A)^2 b\}$ (where $\bar{\Phi}^{-1}x = \{y \mid \lambda y = x$ for some $\lambda \notin \Omega\}$) is an ideal in A containing A^2 and ΩA. Conclude $\bar{A} = A/K$ is trivial as an algebra and torsion-free as $\bar{\Phi}$-module, with d.c.c. on $\bar{\Phi}$-submodules; in particular, it is finitely spanned.
5. Show $A = C+K$ for some finitely generated subalgebra C of A. If A is alternative and locally nilpotent, show $M(C)^n = 0$ for some n. This is the universal n we've been seeking. Show $M(A)^k b \subset M(C)^k b + \bar{\Phi}^{-1} M(A)^{k+1} b$ for any $b \in B$ and $k \geq 0$, then $M(A)^n b \subset \bar{\Phi}^{-1} M(A)^{n+s} b$ for all $s \geq 1$, then $M(A)^n B \subset B \cap \bar{\Phi}^{-1}(0) = B_0$. Conclude $M(A)^{n+1} B = 0$.
6. Deduce <u>Theorem</u>: If A is a locally nilpotent alternative algebra with d.c.c. on ideals, then A is nilpotent.