3. Semiprimeness

A well behaved algebra contains no trivial ideals, which is equivalent to the absence of solvable or nilpotent ideals. This property of semiprimeness is inherited by ideals. We construct a semiprime radical as a measure of non-semiprimeness. In the presence of chain conditions on ideals the semiprime radical is nilpotent, so in this case solvability or nilpotence or antiprimeness of an ideal are all equivalent.

Semiprimeness

Although solvability, Jordan solvability and nilpotence for a particular ideal are not the same, presence or absence of solvability or Jordan solvability is the same as presence or absence of nilpotence:

3.1 (Semiprime Equivalence Theorem). The following are equivalent for an alternative algebra A:

(i) A contains no nonzero nilpotent ideals B, $B^n = 0$

(ii) A contains no nonzero solvable ideals B, $D^n(B) = 0$

(iii) A contains no nonzero Jordan solvable ideals B, $J^n(B) = 0$ or $P^n(B) = 0$

(iv) A contains no nonzero Jordan trivial ideals B, $U_B B = 0$ or $U_B \hat{A} = 0$

(v) A contains no nonzero trivial ideals B, $B^2 = 0$.

Proof. We have seen nilpotence implies solvability, (ii)$\Rightarrow$(i), and solvability implies Jordan-solvability, (iii)$\Rightarrow$(ii). We also noted that if $J^n(B) = 0$ but $C = J^{n-1}(B) \neq 0$ then $J(C) = 0$ (similarly if $P^n(B) = 0$ then $C = P^{n-1}(B)$ has $P(C) = 0$), so (iv)$\Rightarrow$(iii). If $P(B) = U_B\hat{A} = 0$ for an ideal $B \neq 0$ then by Corollary III.1.8 A contains trivial ideals: all Bb are trivial, and if they are all zero then $B^2 = 0$ and B itself is trivial. Thus (v)$\Rightarrow$(iv).

Clearly (i) $\Rightarrow$ (v). ■

In keeping with general terminology, an algebra without trivial ideals could thus be described as trivia-free, nilpotence-free, or solvability-free, but as with associative algebras it is more customary to call such an algebra semiprime. Almost any decent algebra will be semiprime; a trivial algebra has no structure (it is a mere module), so an algebra with a trivial ideal has a worthless part.

It is very important that simple algebras are semiprime: since they are not themselves trivial they contain no trivial ideals, hence are semiprime by 3.1.

3.2 (Semiprimeness Theorem for Simple Algebras) A simple alternative algebra is semiprime. ■

We have seen that every ideal (indeed, every subalgebra) inherits solvability from its parent algebra. It is far more difficult to show semiprimeness is inherited.

3.3 (Semiprime Inheritance Theorem) If A is a semiprime alternative algebra then any ideal $B \lhd A$ is semiprime too.

Proof. The union of an increasing chain of trivial ideals is still trivial, so by Zorn's Lemma we can choose a maximal trivial ideal $C \lhd B$. We claim C is necessarily an ideal of A, $C \lhd A$, which by semiprimeness will force C to be zero and B to be semiprime.

We will check only that C is a right ideal in A. What we will do is prove that $C\hat{A}$ is again a trivial ideal in B; then by maximality of C we will have $C\hat{A} = C$ and $CA \subseteq C$ as desired.

Our tool will be middle annihilation. Recall from III.1.3 that the middle annihilator $D^{\perp,M} = \{x \mid U_D x = 0\}$ is an ideal in A whenever D is, so $E = D \cap D^{\perp,M}$ is a Jordan-trivial ideal of A contained inside B ($U_E E \subset U_D D^{\perp,M} = 0$). For semiprime A we can use 3.1(iv) to conclude $E = D \cap D^{\perp,M} = 0$ for any ideal D:

$$(3.4) \qquad U_D z = 0 \text{ implies } z = 0 \quad (z \in D \lhd A \text{ for } A \text{ semiprime}).$$

As an application we obtain the following associativity result:

$$(3.5) \quad [x,y,D] = 0 \text{ implies } [x,y,A] = 0 \quad ([x,y,A] \subset D \lhd A \text{ semiprime}).$$

Indeed, $U_D[x,y,A] = [x,y,U_D A] - U_{D,[x,y,D]}A$ since associator maps are Jordan derivations by I.3.8, so if $[x,y,D] = 0$ we have $U_D z = 0$ for all $z \in [x,y,A]$, hence by (3.4) $z = 0$ if $z \in D$; if $[x,y,A] \subset D$ we see all z vanish and $[x,y,A] = 0$.

As a first instance of this, for arbitrary $x,y \in C$ and $D = B$ we have $[x,y,B] = 0$ since $[C,C,B] \subset C^2 = 0$ if $C \lhd B$ is trivial; by (3.5) we get $[x,y,A] = 0$ and $[C,C,A] = 0$.

From this $C(C\hat{A}) = 0$, hence $[B,C,C\hat{A}] = 0$ (using $BC \subset C$), and by (3.5) $[C,C\hat{A},B] = 0$ implies $[C,C\hat{A},A] = 0$ (noting $[C,C\hat{A},A] \subset [C,B,A] \subset B$).

On the other hand, from $[C,B,C\hat{A}] = C(C\hat{A}) = 0$ and linearized left bumping we have $[(C\hat{A})C,B,B] = -[BC,B,C\hat{A}]+[C,B,B](C\hat{A})+[C,B,C\hat{A}]B \subset [C,B,C\hat{A}]+C(C\hat{A}) +[C,B,C\hat{A}]B = 0$. Then $[(CA)C,B,B] = 0$ implies $[(CA)C,B,A] = 0$ by (3.5) (using $[(CA)C,B,A] \subset B$), and $[(CA)C,A,B] = 0$ implies $[(CA)C,A,A] = 0$ by (3.5) again (using $[(CA)C,A,A] \subset [B,A,A] \subset B$), so $(C\hat{A})C$ lies in the nucleus of A. Further, its elements are trivial since $(C\hat{A})C \cdot A\{(C\hat{A})C\} = (C\hat{A})C \cdot \{A(C\hat{A})\}C$ (using $[A,C\hat{A},C] = 0) \subset BC \cdot BC \subset C^2 = 0$ (using $C \lhd B \lhd A$). Therefore by III.1.6 the elements of $(C\hat{A})C$ generate trivial ideals, so by semiprimeness they all vanish and

$$(C\hat{A})C = 0.$$

From this we obtain $(C\hat{A})^2 = [(C\hat{A})C]\hat{A} = 0$ (using $[C\hat{A},C,\hat{A}] = 0$), and $C\hat{A}$ is trivial as claimed. ∎

The semiprime radical

We have indicated that most well-behaved algebras are semiprime. Ideally we would like to break an arbitrary algebra into two pieces, a nilpotent piece and a semiprime piece, and analyze the structure of the two pieces separately. The procedure is to construct a radical, a maximal nilpotent ideal R in A such that the quotient A/R is semiprime.

In Chapter VIII we will see that such a procedure works for finite-dimensional alternative algebras: A can be decomposed as the semi-direct sum of its nilpotent piece R and its semiprime piece A/R (if the latter is separable). Without some sort of finiteness condition, however, there is in general no maximal nilpotent ideal.

This technique of isolating an undesirable property in a radical R such that the quotient A/R is uncontaminated by that property, occurs frequently in algebra. There are a wide variety of radicals constructed for different purposes, depending on the undesirable property one is trying to isolate in R or the desirable property one is trying to attain in A/R. We will return to the general theory of radicals in Section 10.

We have indicated that there is in general no maximal nilpotent or solvable ideal, so we define our radical as the smallest ideal we have to divide out by to get rid of nilpotence. First observe that such a smallest ideal exists. There always exist ideals B such that A/B is semiprime (at worst we could take B = A), and the intersection $B = \bigcap B_\alpha$ of all such ideals B_α again has the property that A/B is semiprime: indeed, if C is an ideal in A such that C/B is trivial in A/B,

$C^2 \subset B$, then C/B_α is trivial in A/B_α, $C^2 \subset B \subset B_\alpha$, so by semiprimeness of A/B_α we have $C/B_\alpha = 0$ and $C \subset B_\alpha$ for all α, so $C \subset \bigcap B_\alpha = B$ and $C/B = 0$. Clearly B is the smallest ideal with semiprime quotient.

This smallest ideal B of A such that A/B is semiprime is called the **semiprime radical** S(A) (more commonly but more confusingly it is called the **prime radical**).

3.6 (Semiprime Radical Theorem) Every alternative algebra A contains a unique smallest ideal S(A) such that the quotient A/S(A) is semiprime. ■

3.7 Proposition. A is semiprime iff S(A) = 0. ■

If a largest solvable ideal exists, it coincides with the semiprime radical.

3.8 Proposition. If the alternative algebra A contains a maximal solvable ideal B (e.g., if A has a.c.c. on ideals), then B coincides with the semiprime radical S(A).

Proof. If B is any solvable ideal its image B/S(A) is by 2.6 solvable in the semiprime algebra A/S(A), hence B/S(A) is zero and $B \subset S(A)$. If C/B is solvable in A/B for solvable B then by recoverability 2.6 C is solvable in A, so if B is maximal C = B and A/B has no nonzero solvable ideals; thus A/B is semiprime and $B \supset S(A)$ for maximal solvable B. ■

3.9 <u>Remark</u>. A maximal nilpotent ideal (if such exists) need not coincide with the semiprime radical. The above proof breaks down because of the non-recoverability of nilpotence (see 2.10). ■

So far all we know about the semiprime radical is that it is the bare minimum we must jettison to reach semiprimeness. If B is an ideal in A which is trivial, nilpotent, solvable, or Jordan-solvable then so is its image in the semiprime algebra $A/S(A)$, so by 3.1 this image must be zero: $S(A)$ contains all nilpotent, solvable, or Jordan solvable ideals of A.

Semiprime algebras are ones whose semiprime radicals vanish, $S(A) = 0$. At the opposite extreme are the antiprime (also S-radical or Baer-radical) algebras, those which are all radical:

$$S(A) = A.$$

By definition this means that in order to attain semiprimeness we must get rid of the whole algebra, as all nonzero quotients are still contaminated by nilpotence, and the only semiprime quotient or image of A is zero: an algebra is antiprime iff it has no nonzero semiprime images.

For example, a nonzero semiprime algebra has itself as nonzero semiprime image, so

(3.10) A semiprime and antiprime implies $A = 0$.

This can also be seen by noting $S(A) = 0$ and $S(A) = A$ imply $A = 0$. By Semiprime Inheritance 3.3, any ideal B in a semiprime A is again semiprime, so the only antiprime ideal in A is $B = 0$. In fact (ex. 3.15), an alternative algebra is semiprime iff it contains no antiprime ideals.

Any homomorphic image $\bar{A}$ of an antiprime algebra A is again antiprime, since any nonzero semiprime image of $\bar{A}$ is at the same time a nonzero semiprime image of A.

It is also true (Ex. 3.17) that any ideal in an antiprime ideal is antiprime, and (Ex. 3.16) $S(A)$ is precisely the maximal antiprime ideal of A.

Algebras with chain conditions

Solvability and nilpotence coincide in the presence of the a.c.c. or d.c.c. on ideals. Indeed, in these cases the semiprime radical turns out to be nilpotent.

In general a trivial ideal need not act trivially on a bimodule; we begin by finding conditions under which a trivial ideal acts nilpotently. First we have a general construction.

3.11 Lemma. If $B \lhd A$ is a trivial ideal, $B^2 = 0$, and N an A-bimodule, then for any $b \in B$

$$N_b = b(BN) = B(bN)$$

is a sub-A-bimodule with $bN_b = 0$, and $L_B^k N_b = 0 \Leftrightarrow L_b L_B^{k+1} N = 0$.

Proof. The two expressions for N_b coincide since $L_b L_B + L_B L_b = L_{b \circ B}$ where $b \circ B \subset B^2 = 0$ by hypothesis. This N_b is left-invariant under A because $AN_b = L_A L_B L_b N = \{-L_b L_B L_A + L_{U(b,A)B}\}N$ (Left Moufang) $\subset L_b L_B N \subset N_b$ (since $U_{b,A}B \subset B^2 = 0$ if $B \lhd A$). It is right invariant because $N_b A = \{b(BN)\}A$ $= L_{b(BN)}A = \{-L_{N(Bb)} + L_b L_B L_N + L_N L_B L_b\}A$ (Left Moufang) $\subset L_b L_B N = N_b$ (since Bb and $L_B L_b A$ are contained in $B^2 = 0$). We have $bN_b = b^2(BN) = 0$ since $b^2 = 0$, and $L_B^k N_b = 0 \Leftrightarrow L_B^k(L_B L_b N) = 0 \Leftrightarrow L_b L_B^{k+1} N = 0$ (recall $L_b L_c = -L_c L_b$). ∎

From this we can show a trivial id#al acts nilpotently in the presence of chain conditions.

3.12 Lemma. If A has a.c.c. on ideals or its bimodule M has d.c.c. on submodules, then any trivial ideal $B \lhd A$ acts nilpotently on M: $M(B)^n_M = 0$ for some n.

Proof. If B does not act nilpotently on M then it can't act both left-nilpotently and right-nilpotently, by Left-Right Nilpotence 1.16; let us assume B is not left-nilpotent on M.

First consider the d.c.c. case. Choose a submodule $N \triangleleft M$ minimal among those on which B acts non-left-nilpotently. By III.1.11 $L_B N$ is a submodule contained in N on which B does not act left nilpotently (since $L_B^n(L_B N) = L_B^{n+1} N \neq 0$ if B is not left nilpotent on N), so by minimality $L_B N = N$. Then $L_B^k N = N$ for all k. We can choose b so $N_b = L_b N \neq 0$. By Lemma 3.11 $N_b \triangleleft N \triangleleft M$ is a submodule contained in N on which B doesn't act left nilpotently, $L_b L_B^{k+1} N = L_b N \neq 0 \Rightarrow L_B^k N_b \neq 0$, yet $L_b N_b = 0$ so $N_b \neq N$ and $N_b < N$. But this contradicts the minimality of N. Thus non-nilpotence leads to a contradiction.

Now suppose A has a.c.c.. For any submodule $N \triangleleft M$ the left annihilators $Ann_L(N) \subset Ann_L(L_B N) \subset \cdots \subset Ann_L(L_B^k N)$ form an increasing chain of ideals in A, by III.1.11, so terminate by a.c.c. at some $Ann_L(L_B^k N)$, which we shall denote by $Ann_\infty(N)$. Non-left-nilpotence means $B \not\subset Ann_\infty(M)$, so by the a.c.c. we can choose an annihilator $Ann_\infty(N)$ maximal among those not containing B. Then some $b \notin B$ doesn't belong to $Ann_\infty(N)$; by Lemma 3.11 $N_b = L_b L_B N$ is a submodule contained in N, so $Ann_\infty(N) \subseteq Ann_\infty(N_b)$, and in fact the inclusion is strict since $b \in Ann_L(N_b) \subset Ann_\infty(N_b)$ but $b \notin Ann_\infty(N)$. Again by Lemma 3.11 $b \notin Ann_L(L_B^{k+2} N) \Rightarrow L_b L_B^{k+2} N \neq 0 \Rightarrow L_B^{k+1} N_b \neq 0 \Rightarrow B \not\subset Ann_L(L_B^k N) \rightarrow B \not\subset Ann_\infty(N)$. But $Ann_\infty(N) < Ann_\infty(N_b)$ and $B \not\subset Ann_\infty(N_b)$ contradicts the maximality of $Ann_\infty(N)$. Thus non-nilpotence of B leads to a contradiction. ∎

We can strengthen this to the case where D(B) acts nilpotently, not merely where D(B) is zero.

3.13 **Lemma.** Let A have a.c.c. on ideals or its bimodule M have d.c.c. on submodules, and let $B \triangleleft A$ be an ideal. Then if the derived algebra $D(B) = B^2$ acts nilpotently on M, so does B.

Proof. By nilpotence $M(B^2)^{n+1}M = 0$ we have by III.1.11 a chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_{n+1} = 0$ for $M_k = M(B^2)^k M$. If we can find integers $f(k)$ such that $M(B)^{f(k)} M_k \subset M_{k+1}$ then we will have $M(B)^{f(n)+\cdots+f(0)}M = M(B)^{f(n)} (\cdots M(B)^{f(0)} M_0) \subset M(B)^{f(n)} (\cdots M(B)^{f(1)} M_1) \subset \cdots \subset M(B)^{f(n)} M_n \subset M_{n+1} = 0$ and $M(B)$ will act nilpotently. Thus we only need to be able to boost M_k into M_{k+1}.

Finding the $f(k)$ with $M(B)^{f(k)} M_k \subset M_{k+1}$ amounts to finding integers such that $M(\bar{B})^{f(k)} \bar{M}_k = \bar{0}$ on $\bar{M}_k = M_k/M_{k+1}$, which is a bimodule over $\bar{A} = A/B^2$ by III.1.11 since $M(B^2)M_k = M_{k+1}$ implies $M(B^2)\bar{M}_k = 0$ and $B^2 \subset \mathrm{Ann}(\bar{M}_k)$. Now $\bar{A}$ inherits the a.c.c. from A or $\bar{M}_k$ inherits the d.c.c. from M_k (and in turn from M), and $\bar{B}$ is a trivial ideal in $\bar{A}$, so by the previous lemma $M(\bar{B})$ is nilpotent on $\bar{M}_k$: $M(\bar{B})^{f(k)} \bar{M}_k = 0$ for some $f(k)$ as desired. ∎

In terms of the regular bimodule this becomes the

3.14 (Zhevlakov-Slater Nilpotence Theorem) If A is an alternative algebra with a.c.c. or d.c.c. on two-sided ideals, then an ideal in A is solvable iff it is nilpotent, and the semiprime radical $S(A)$ is the largest nilpotent ideal, containing all other nilpotent ideals.

If M is an A-bimodule where A has a.c.c. on ideals or M has d.c.c. on subbimodules (whether A has d.c.c. or not), then any solvable ideal in A acts nilpotently on M.

Proof. If A has a.c.c. or M has d.c.c. then by Lemma 3.13 whenever $D(B)$ acts nilpotently so does the ideal B, hence by induction whenever $D^n(B)$ acts nilpotently so does B, so in particular if B is solvable ($D^n(B) = 0$) it acts nilpotently on M.

Applying this to the regular bimodule M = A we see that if A has a.c.c. or d.c.c. on ideals then all solvable ideals act nilpotently on A, so an ideal is solvable iff it is nilpotent. The a.c.c. case is now easy: in the presence of the a.c.c. S(A) is by 3.3 the unique maximal solvable ideal, which by our remarks is nilpotent.

Now consider the d.c.c. case. In the presence of the d.c.c. the descending chain $S(A) \supset D(S(A)) \supset D^2(S(A)) \supset \cdots$ must terminate at $B = D^n(S(A)) = D^{n+1}(S(A)) = \cdots$ with $D(B) = B$. We will show $B = 0$, so S(A) is solvable (therefore by our remarks nilpotent).

It is a general fact (independent of d.c.c.) that S(A) annihilates all minimal ideals $C \triangleleft A$. Indeed, a minimal ideal C is an irreducible $\hat{A}$-bimodule, so rather trivially has d.c.c. on sub-bimodules. Therefore if D is an ideal in A such that D^2 acts trivially ($M(D^2)C = 0$) we know by 3.13 that D itself acts nilpotently on C. But if $M(D)^n C = 0$ we can't have $M(D)C = C$, yet $M(D)C$ is by III.1.1 an ideal contained in C, so by minimality $M(D)C = 0$ and $D \subset C^{\perp}$. Thus $D^2 \subset C^{\perp} \Rightarrow D \subset C^{\perp}$, so $A/C^{\perp}$ has no trivial ideals. By construction of the semiprime radical S(A), $A/C^{\perp}$ semiprime implies $C^{\perp} \supset S(A)$, so S(A) annihilates C as claimed.

Thus the semiprime radical $S(\bar{A})$ of $\bar{A} = A/B$ annihilates all minimal ideals $\bar{C}$ of $\bar{A}$, and if $\bar{A}$ is nonzero it has lots of nonzero minimal ideals $\bar{C} \neq 0$ since it inherits the d.c.c. from A. On the other hand, $\bar{B}$ doesn't annihilate anything: $\bar{x} \in \bar{B}^{\perp} \Leftrightarrow M(\bar{B})\bar{x} = \bar{0} \Leftrightarrow M(B)x \subset B^{\perp} \Leftrightarrow M(B)M(B)x = 0 \Rightarrow M(B^2)x = 0$ (by 1.7) $\Rightarrow M(B)x = 0$ ($B^2 = B$ by choice of B) $\Rightarrow x \in B^{\perp} \Rightarrow \bar{x} = \bar{0}$. Yet $\bar{B} \subset S(\bar{A})$ because if D denotes the preimage of $S(\bar{A})$ in A the fact that $A/D \cong (A/B^{\perp})/(D/B^{\perp})$ $= \bar{A}/S(\bar{A})$ is semiprime implies $D \supset S(A)$ by construction, therefore $B \subset S(A) \subset D$ and $\bar{B} \subset \bar{D} = S(\bar{A})$. The only way out of this apparent contradiction is for $\bar{A}$ to be zero, $A = B^{\perp}$. Then $B = B^2 = M(B)B \subset M(B)B^{\perp} = 0$ and S(A) is solvable. ∎

Exercises IV.3

3.1 In the Semiprime Equivalence Theorem, take any two conditions and prove directly that one implies the other.

3.2 We could define an algebra to be Jordan-semiprime if it contains no (nonzero) Jordan-solvable Jordan-ideals. Prove the Jordan Semiprimeness Theorem : An alternative algebra is Jordan-semiprime iff it is semiprime.

3.3 Show that A is semiprime iff $B \cap B^{\perp} = 0$ for all ideals $B \triangleleft A$.

3.4 If $C \triangleleft B \triangleleft A$ where C is trivial and $B \cap \mathrm{Ann}_L(B) = 0$ (as when A is semiprime), show $C(CA) = 0$ by showing $\{C(CA)\}B = 0$.

3.5 In the Semiprime Inheritance Theorem show that $(C\hat{A})C$ is spanned by trivial nuclear elements $(da)c$ by using the Fundamental Formulas.

3.6 If $z \in N(B)$ show $[z,U_B B,A] = 0$. Conclude $z \cdot U_B B$ is a right ideal in A. If B is semiprime and $z \in C \cap N(B)$ for $C \triangleleft B$ trivial, conclude $z \cdot U_B B = 0$; deduce $U_B z = 0$ and $z = 0$, so $C \cap N(B) = 0$. Use this to give an alternate proof of $(C\hat{A})C = 0$.

3.7 Given $C(C\hat{A}) = (C\hat{A})C = 0$, show $B(C\hat{A})^2 = 0$, so that $(C\hat{A})^2 = 0$ when $B \cap \mathrm{Ann}_R(B) = 0$.

3.8 Prove CA is trivial in Semiprime Inheritance by showing (i) $U_B[C,C,A] = 0$, (ii) $[C,C,A] = C(C\hat{A}) = 0$, (iii) $[B,C,C\hat{A}] = 0$, (iv) $[(C\hat{A})C,B,B] = 0$, (v) $[(C\hat{A})C,B,A] = 0$, (vi) $\{(C\hat{A})C\}B$ is a trivial right ideal in A, (vii) $(C\hat{A})C = 0$, (viii) $U_{B^2}(C\hat{A})^2 = 0$ using V.0.0.

3.9 Show that Semiprime Inheritance holds if A is merely assumed to be B-semiprime in the sense that there are no trivial ideals of A which lie wholly inside B.

3.10 Show that any subdirect sum of semiprime algebras is semiprime (see 0.00 for the definition), and use this to establish the existence of S(A) in 3.1.

3.11 Although a Jordan-solvable ideal B need not be solvable, show it is at least antiprime: S(B) = B.

3.12 To see why we use the term "antiprime" instead of "antisemiprime", show A has no semiprime images iff it has no prime images. (Hint: given $x \neq 0$ choose a sequence $x_1, x_2, \cdots$ with $x_1 = x$ and $0 \neq x_{n+1} \in I(x_n)^2$; if B is an ideal maximal with respect to $B \cap \{x_1, x_2, \cdots\} = \emptyset$, then A/B is prime).

3.13 By first removing all nilpotent ideals from A, then removing all nilpotent ideals from the remaining quotient algebra, and repeating until there are no nilpotent ideals left, we eventually reach the semiprime radical. Prove this <u>Inductive Construction of the Semiprime Radical</u>: If A is an alternative algebra we define an increasing chain of ideals $S_\lambda(A)$ for each ordinal number λ by setting $S_0(A) = 0$, $S_\lambda(A)$ = the sum of all ideals of A which are trivial (resp. nilpotent, solvable) modulo $S_\mu(A)$ if $\lambda = \mu+1$ is a successor, and $S_\lambda(A) = \sum_{\mu<\lambda} S_\mu(A) = \bigcup_{\mu<\lambda} S_\mu(A)$ if λ is a limit ordinal. Then the chain breaks off at the semiprime radical, $S_\lambda(A) = S(A)$ for $||\lambda|| > |A|$ ($||\lambda||$ = cardinality of the segment $\{\mu | \mu < \lambda\}$).

3.14 Use the Inductive Construction of S(A) to show $B \cap S(A) \subset S(B)$ for all $B \triangleleft A$. Deduce S(A) is antiprime for any alternative algebra,

$$S(S(A)) = 0.$$

3.15 Show A is semiprime iff it contains no antiprime ideals.

3.16 Use Semiprime Inheritance to prove the <u>Antiprime Radical Theorem</u>: The semiprime radical S(A) of an alternative algebra A is the maximum antiprime ideal of A, containing all other antiprime ideals. The quotient A/S(A) has no antiprime ideals.

3.17 Prove the Antiprime Inheritance Theorem: Any ideal B in an antiprime algebra $S(A) = A$ is again antiprime, $S(B) = B$.

3.18 Prove the Semiprime Radical Inheritance Theorem: The semiprime radical of an ideal $B \lhd A$ is

$$S(B) = B \cap S(A).$$

Deduce $B \cap S(A) \supset S(B)$ from Semiprime Inheritance. Deduce $B \cap S(A) \subset S(B)$ from Antiprime Inheritance.

3.19 Prove the Lemma: If M is an A-bimodule with a.c.c. on right annihilator submodules $\mathrm{Ann}(S) = \{m \in M \mid Sm = 0\}$ for $S \subset L(A)$, and $B \lhd A$ has $M(B^2)M = 0$, show B acts left nilpotently on M.

3.20 Prove the Lemma: If M is an A-bimodule with a.c.c. on left and right annihilator submodules, show that any solvable ideal $B \lhd A$ acts nilpotently on M.

3.21 Let $S = \{s\}$ be a collection of linear transformations on a module M such that

$$s^2 = 0, \quad st = -ts$$

for all $s,t \in S$. Show if M has a.c.c. on S-invariant subspaces (or just annihilator subspaces) then S acts nilpotently on M.

3.22 If $B \lhd A$, $N \lhd M$ show $N_b = b(BN) + M(B^2)N \lhd M$ for all $b \in B$, with $bN_b \subset M(B^2)N$. If $M(B)N = N$ show $N_b \subset M(B)^n(bN) + M(B^2)N$ for all $n \geq 1$, and also $N_b = bN + M(B^2)N$.

3.23 Prove the Lemma: If C is a minimal ideal in A which is contained in the prime radical, then $M(S(A))C = 0$.

IV.3.1 Problem Set on Zhevlakov's Original Proof

1. If $B \triangleleft A$, $N \triangleleft M$ where $B^2N = 0$, show all $N_b = b(BN)$ are left A-submodules with $N_b \subset N$, $bN_b = 0$. If M has <u>d.c.c. on left A-submodules</u> and $B^2M = 0$, show B acts left-nilpotently on M. (We need left d.c.c. because the N_b are not in general sub-bimodules if we don't assume $MB^2 = 0$). Prove that in the presence of the left d.c.c., if D(B) acts left nilpotently so does B. Conclude that every solvable ideal B acts left nilpotently.
2. Prove that if M has d.c.c. on left submodule and $B \triangleleft A$ there are integers Z(n) such that $L_B^{Z(n)}M \subset L_{D^n(B)}M$. In particular, there is m such that $2L_B^m M \subset L_{P^+(B)}M$.
3. Show that if $B \triangleleft A$ kills all minimal ideals $C \subset B$, it kills all minimal ideals of A whatsoever.
4. If A has d.c.c. on left submodules contained in N, where N is a minimal sub-bimodule of M, show S(A) kills N from the left.
5. Show that if S(A) kills a minimal sub-bimodule N from the left then it also kills N from the right. Here no chain conditions are needed. (Again, Zhevlakov chose a minimal $\overline{B}$ in $\overline{S(A)}$, but this can be avoided).
6. Prove the <u>Lemma</u>. If M has d.c.c. on left submodules then S(A) kills any minimal sub-bimodule $N \triangleleft M$.
7. Prove that if $B, C, D \triangleleft A$ with $M(B)C \subset D^{\perp,L}$ then $L(C)L(P^+(B))D = 0$. Conclude that if A has d.c.c. on left ideals and $B, C, D \triangleleft A$ with $M(C)B \subset D^{\perp,L}$ then for some m $2L(C)L(B)^m D = 0$.
8. Prove <u>Theorem</u>. If A has d.c.c. on left ideals but no 2-torsion, then S(A) is nilpotent.