§4 Associativity from commutativity.

The Nucleus=Center Theorem says that in nice algebras which aren't associative, anything that associates with A also commutes with A. Conversely, commutativity goes a long way towards implying associativity: in characteristic $\neq 3$ situations, anything which commutes with A also associates with A. Characteristic 3 is the bad characteristic for alternative algebras; in this section the coefficient 3 will crop up in lots of unwanted places (recall also 2.1).

4.1 (Commutativity-Implies-Associativity Lemma). If z commutes with A then 3z and z^3 lie in the center of A, and $[x,y,z]^2 = 0$ for all x,y. If A has no nilpotent elements, or if 3 is injective or surjective, then any element which commutes with A lies in the center.

Proof. If z commutes with everything so does 3z, and moreover 3z associates with everything by the first of

(4.2) $3[x,y,z] = [xy,z] - x[y,z] - [x,z]y$ (Associator -

(4.3) $3[x,y,z] = [xy,z] + [yz,x] + [zx,y]$. Commutator Formulas)

These are just formulas (2.8) and (2.9).

By Artin's Theorem, if z commutes with all x so does z^3, and z^3 associates since $[z^3,x,y] = (z^3x)y - z^3(xy) = \{z(zx)z\}y - z\{z^2(xy)\} = z\{(zx)(zy) - z(z(xy))\}$ (left Moufang) $= z\{(zx)(yz) - z(xy)z\} = 0$ (middle Moufang).

For $[x,y,z]^2 = 0$ we need

$$[z,y,\ [z,a,y]] = 0$$

for all $a,y \in A$. This follows from $(zy)[z,a,y] = (zy)\{(za)y\} - (zy)\{z(ay)\} = \{yz\}\{(za)y\} - \{zy\}\{(ay)z\} = U_y\ z(za) - U_z\ y(ay)$ (middle Moufang) and $z\{y[z,a,y]\} = z[y,z,ay]$ (left bumping) $= - z[y,ay,z] = - z\{[(ya)y]z\} - y[(ay)z]\} = - U_z(yay) + z\{y[z(ay)]\} = - U_z\ y(ay) + (zyz)(ay)$ (left Moufang) $= - U_z\ y(ay) + U_y\ z^2a$. Setting $a = x^2$ gives $0 = [z,y,[z,y,x^2]] = [z,y,x\circ[z,y,x]]$ (middle bumping) $= x\circ[z,y,[z,y,x]] + [z,y,x]\circ[z,y,x]$ (linearized middle bumping) $= 0 + 2[z,y,x]^2$; since already $3[z,y,x]^2 = 0$ from $3[z,y,x] = 0$, we see $[z,y,x]^2 = 0$.

If A has no nilpotent elements then $[z,y,x]^2 = 0$ implies $[z,y,x] = 0$, and z is nuclear and so central. Similarly, if 3 is injective $3[z,y,x] = 0$ implies $[z,y,x] = 0$ and z is central. If 3 is surjective, $3A = A$, then $[z,A,A] = 3[z,A,A] = 0$ again implies z is central. ☐

4.4 Corollary. A commutative alternative division algebra is a (commutative, associative) field. ☐

The Corollary has a geometric significance. Commutativity of the coordinate ring of a plane corresponds to Pappus' Theorem, associativity to Desargues' Theorem. The Corollary says that Pappus implies Desargues (a result which can also be proven geometrically; see Appendix V).

The Corollary actually holds for simple algebras, but the proof at one point relies on results about polynomal identities; it would be nice to have an elementary proof.

When not just z but all of A commutes, it is relatively easy to show $[x,y,z]^3 = 0$: writing $[x,y,z] = u - v$ for $u = (xy)z$, $v = x(yz)$ we have $[x,y,z]^3 = (u - v)^3$ $= u^3 - v^3 - 3uv(u - v)$ by Artin's Theorem since $\Phi[u,v]$ is commutative associative; but $3(u - v) = 3[x,y,z] = 0$, and using Artin's Theorem twice $u^3 = \{(xy)z\}^3 = (xy)^3 z^3 = (x^3y^3)z^3$, $v^3 = x^3(y^3z^3)$, so also $u^3 - v^3 = [x^3,y^3,z^3] = 0$. Thus $[x,y,z]^3 = 0$.

We give examples to show that in characteristic 3 commutativity is not enough to imply associativity.

4.5 Example. We construct a commutative alternative algebra, of dimension 6 over any ring of characteristic 3, which is not associative. It has multiplication table

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	x_4	0	0	x_6	0
x_2	x_4	0	x_5	0	0	0
x_3	0	x_5	0	$-x_6$	0	0
x_4	0	0	$-x_6$	0	0	0
x_5	x_6	0	0	0	0	0
x_6	0	0	0	0	0	0

By symmetry of the table, A is commutative. Therefore we need only check left alternativity. Since for $i \geq 4$ $Ax_i + x_i A \subseteq \Phi x_6$ and $Ax_6 = x_6 A = 0$, $[A,A,x_i] = [A,x_i,A] = [x_i,A,A] = 0$ trivially for $i \geq 4$. Thus we need only consider associators involving only x_1, x_2, x_3. Again $[x_i,x_i,A] = 0$ is trivial since $x_i^2 = 0$ and $x_i(x_i A) = 0$. Thus we need only $[x,y,z] + [y,x,z] = 0$ for x,y,z basis elements x_i $(1 \leq i \leq 3)$. Here $[x_1,x_2,x_3] + [x_2,x_1,x_3]$ $= 2x_4x_3 - x_1x_3 = -3x_6 = 0$ in characteristic 3, $[x_1,x_3,x_2]$ $+ [x_3,x_1,x_2] = -x_1x_5 - x_3x_4 = -x_6 + x_6 = 0$, $[x_2,x_3,x_1]$ $+ [x_3,x_2,x_1] = 2x_5x_1 = x_3x_4 = 3x_6 = 0$ because of characteristic 3 again.

Thus A is alternative. However, it is not associative since $[x_1,x_2,x_3] = x_4x_3 - x_1x_5 = -x_6 - x_6 = -2x_6 \neq 0$.

A is nilpotent, $A^4 = 0$, but the intersection of all nonzero ideals (the "heart" of A) turns out to be Φx_6. ∎

4.6 Example. Let $\Phi[x,y,z]$ be the free alternative algebra on 3 generators over a field Φ of characteristic 3, and $A = \Phi[x,y,z]/K$ for K the ideal generated by $[x,y],[y,z],[z,x]$. In characteristic 3, (4.2) shows any $a \to [a,b]$ is a derivation; thus if the commutators of degree 2 in the generators vanish, all commutators vanish since they can be broken down into degree 2 commutators, and A is commutative. It is not associative since $[x,y,z] \not\in K$, $[x,y,z]$ cannot be generated by $[x,y],[y,z],[z,x]$. Indeed, if it could then by considering degrees in x,y,z there would have to be a relation $\alpha_1 z[x,y] + \beta_1[x,y]z + \alpha_2 x[y,z] + \beta_2[y,z]x + \alpha_3 y[z,x] + \beta_3[z,x]y = [x,y,z]$ back in $\Phi[x,y,z]$. But then the same would hold in the free associative algebra. Expanding the commutators out, the coefficients of xyz,xzy,yzx,yxz,zxy,zyx are respectively $\beta_1 + \alpha_2$, $-\alpha_2 - \beta_3$, $\beta_2 + \alpha_3$, $-\alpha_3 - \beta_1$, $\alpha_1 + \beta_3$, $-\alpha_1 - \beta_2$; these all must vanish ($[x,y,z] = 0$ in the free associative algebra), so $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, $\beta_1 = \beta_2 = \beta_3 = -\alpha$. Going back to the free alternative, we must have $[x,y,z] = \alpha\{[z,[x,y]] + [x,[y,z]] + [y,[z,x]]\} = 6\alpha[x,y,z] = 0$ in characteristic 3, whereas $[x,y,z] \neq 0$ in $\Phi[x,y,z]$. (We have an epimorphism onto the Cayley algebra via $x \to e_{12}^{(1)}$, $y \to e_{12}^{(2)}$, $z \to e_{12}^{(3)}$, $[x,y,z] \to e_{22} - e_{11} \neq 0$) . Thus $[x,y,z] \notin K$ and $[\bar{x},\bar{y},\bar{z}] \neq 0$ in A .

We can modify this to give an example promised earlier of an isotope which is not isomorphic.

4.7 Example. Set $A' = \Phi[x,y,z]/K'$ where K' is generated by K and z^3 (so we take the commutative alternative algebra A and divide out by the relation $\bar{z}^3 = 0$). This new A' inherits commutativity, and by degree considerations $[x,y,z]$ still isn't in K' (i.e. $[\bar{x},\bar{y},\bar{z}] \neq 0$ in A') . Since $\bar{z}$ has been rendered nilpotent, $\bar{u} = 1 - \bar{z}$ has been rendered invertible. We claim the isotope $A^{(u,1)}$ is not even commutative, so can't possibly be isomorphic to A: we have $[\bar{x},\bar{y}]^{(u,1)} = \bar{x} \cdot_{u,1} \bar{y} - \bar{y} \cdot_{u,1} \bar{x}$
$= (\bar{x}\bar{u})\bar{y} - (\bar{y}\bar{u})\bar{x} = (\bar{x}\bar{u})\bar{y} - \bar{x}(\bar{u}\bar{y})$ (commutativity of A) $= [\bar{x}, 1 - \bar{z}, \bar{y}]$
$= - [\bar{x},\bar{z},\bar{y}] \neq 0$.

#5. Problem Set on Casimir Operators

1. If (ℓ, r) is a birepresentation of an alternative algebra A on a finite-dimensional bimodule M over a field Φ, show the <u>left trace form</u> $\tau_\ell(x) = \operatorname{tr} \ell_x$ satisfies $\tau(xy) = \operatorname{tr} \ell_x \ell_y = \operatorname{tr} \ell_y \ell_x = \tau(yx)$ and $\tau((xy)z) = \tau(x(yz))$, so $\tau(x,y) = \tau(yx)$ defines a symmetric, associative bilinear form on A. Show $\tau(z) = 0$ if z is nilpotent and $\tau(e) = \dim_\Phi eM$ if e is idempotent.
2. Show $\operatorname{Ker} \ell = \{x \mid \ell_x = 0\}$ is contained in the radical Rad τ of the bilinear form τ. If Φ has characteristic zero show Rad τ is nil modulo the ideal Ker ℓ. If all ideals $B \triangleleft A$ have units e (as will be the case when A is semisimple with d.c.c. - see Chapter VIII) show Rad τ = Ker ℓ.
3. The <u>Centroid</u> $\Gamma(M)$ of a bimodule M is the set of all linear transformations on M which commute with all L_x, R_x for $x \in A$ (i.e. with $M(A|M)$). Show any $\tilde{T}$ on $E = A \oplus M$ which leaves M invariant and commutes with $M_E(A)$ restricts to an element $T = \tilde{T}|_M$ in $\Gamma(M)$.
4. If M is irreducible, show $\Gamma(M)$ is a division algebra.
5. If (ℓ, r) is a birepresentation of a finite-dimensional algebra A over a field Φ on a finite-dimensional bimodule M with nondegenerate trace form $\tau = \tau_\ell$, show ℓ is faithful; if $\{x_i\}$, $\{x_i^*\}$ are dual basis for A relative to τ, show the <u>left Casimir operator</u> $C_\ell = \Sigma\, \ell_{x_i} \ell_{x_i^*}$ belongs to the centroid $\Gamma(M)$. Show $\operatorname{tr} C_\ell = n = \dim A$, so if the characteristic $\neq 0$ then $C_\ell \neq 0$ and C_ℓ is invertible if M is irreducible.

Unfortunately, the Casimir operator doesn't seem to lead (as it does in the associative or Lie case) to a proof that the cohomology groups of a separable algebra are zero.